Chapter overview

- \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) defined by \( f(x,y) = 2x - 3y \) is example of a linear function, \( g(x,y) = x^2 - 5y \) is not.
- In this chapter, study more generally linear transformations \( T : \mathbb{R}^m \rightarrow \mathbb{R}^n \).
- Even more gen, study linear \( T : V \rightarrow W \) where \( V, W = \text{vector spaces} / \mathbb{F} \). Recall \( V \) is the domain of \( T \) & \( W \) the codomain of \( T \).
- We’ll generalise systems of linear equations \( Ax = b \) to “linear equations” of form \( T x = b \) where \( b \in W, x \in V \).

Often abbrev \( T(x) = Tx \).
Addition Condition

To define linear map, first consider

**Addition Condition.**

We say $T : V \rightarrow W$ satisfies the **addition condition**, if

$$T(v + v') = T(v) + T(v') \text{ for all } v, v' \in V.$$ 

**E.g.** $T : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = 2x - 3y$ satisfies the addn condn since given $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \in \mathbb{R}^2$

$$T \left( \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix} \right) =$$

$$T \begin{pmatrix} x \\ y \end{pmatrix} + T \begin{pmatrix} x' \\ y' \end{pmatrix} =$$

**Warning** The $+$ on the two sides of the equation are different!
Scalar Multiplication Condition

We say \( T : V \rightarrow W \) satisfies the \textbf{scalar multiplication condition}, if

\[
T(\lambda v) = \lambda T(v) \quad \text{for all } \lambda \in \mathbb{F} \text{ and } v \in V.
\]

\textbf{E.g.} \( T : \mathbb{R}^2 \rightarrow \mathbb{R} \) defined by \( T\begin{pmatrix} x \\ y \end{pmatrix} = 2x - 3y \) satisfies the scalar multn condn since given \( \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \lambda \in \mathbb{R} \)

\[
T \left( \lambda \begin{pmatrix} x \\ y \end{pmatrix} \right) = \quad \lambda T \begin{pmatrix} x \\ y \end{pmatrix} =
\]

\textbf{Warning} The scalar multn on the two sides of the eqn are different!
Linear Transformation

**Definition**

Let $V, W = \text{vector spaces} / \mathbb{F}$. A function $T : V \to W$ is called a **linear map** or a **linear transformation** if following both hold.

**Addition Condition.**

$T(v + v') = T(v) + T(v')$ for all $v, v' \in V$, and

**Scalar Multiplication Condition.**

$T(\lambda v) = \lambda T(v)$ for all $\lambda \in \mathbb{F}$ and $v \in V$.

**E.g.** $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = 2x - 3y$ is linear.
Sample question: showing a function is linear.

Example

Show that the function $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by

$$T(x) = \begin{pmatrix} 4x_2 - 3x_3 \\ x_1 + 2x_2 \end{pmatrix} \quad \text{for} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$$

is a linear map.

Solution
Proposition.

If $T : V \rightarrow W$ is a linear map, then $T(0) = 0$.

Proof. $T(0) = T(00) = 0 \cdot T(0) = 0$.

Example

Show that the function $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 - 2 \\ x_1 \end{pmatrix}$ is not linear.

Soln
Another non-linear example

Example

Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2^2 \end{pmatrix}$$

is not linear.

Solution

Checking $T(0) = 0$ here tells you nothing about linearity. Suffice check that $T(\lambda \mathbf{v}) = \lambda T \mathbf{v}$ fails for single choice of pair $\lambda, \mathbf{v}$. 
Alternate characterisation of linearity

We can combine the addn condn & scalar multn condn into one!

**Theorem**

The function $T : V \rightarrow W$ is a linear map iff for all $\lambda \in \mathbb{F}$ and $v_1, v_2 \in V$

$$T(\lambda v_1 + v_2) = \lambda T(v_1) + T(v_2).$$

**Remark** This means that a linear map $T$ has the special property that it sends the line $x = a + \lambda v$ to the line $x = T a + \lambda T v$ or point $T a$ if $T v = 0$.

**E.g.** Differentation is a linear map. More precisely, we define $T : \mathbb{P} \rightarrow \mathbb{P}$ by $Tp = \frac{dp}{dx}$. Then for $p, q \in \mathbb{P}, \lambda \in \mathbb{R}$

$$T(\lambda p + q) =$$
Theorem

If $T : V \to W$ is linear map, $v_1, \ldots, v_n \in V$ & $\lambda_1, \ldots, \lambda_n$ are scalars, then

$$T(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_1 T(v_1) + \cdots + \lambda_n T(v_n).$$

Example

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ a function such that

$$T \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad T \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$ 

Show that $T$ is not linear.

Solution

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7.1 Introduction to Linear Maps
Example

Given that $T$ is a linear map and

$T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$

Find $T \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$

Solution
Linear maps are determined by the values on a spanning set

Previous eg illustrates the following general result.

**Theorem**

Let $T : V \rightarrow W$ be linear & $V = \text{span}(v_1, \ldots, v_m)$. Then $T$ is completely determined by the $m$ values $T v_1, \ldots, T v_m$.

Compare with the following:

An affine linear function $f(x) = mx + b$ is determined by two of its values $f(x_1), f(x_2)$, since its graph is a line which is determined by two points.
Matrices define Linear Maps

**Theorem**

For each $m \times n$ matrix $A$, the function $T_A : \mathbb{R}^n \to \mathbb{R}^m$, defined by

$$T_A(x) = Ax \quad \text{for} \quad x \in \mathbb{R}^n,$$

is a linear map called the **associated linear map**.

**Proof.**
Example of reflection

Example

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, describe the associated linear map $T_A$ geometrically as a mapping $\mathbb{R}^2 \rightarrow \mathbb{R}^2$.

Solution
Matrix Representation Theorem

Conversely, given linear $T : \mathbb{R}^n \to \mathbb{R}^m$, we can find an $m \times n$ matrix $A$ such that $T(x) = Ax$ for all $x \in \mathbb{R}^n$. In this case, we say $A$ is a matrix representing $T$.

Example

Given that $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x - y \\ y \end{pmatrix}$ is linear. Find the matrix $A$ representing $T$.

Solution
Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let the vectors $e_j$ for $1 \leq j \leq n$ be the standard basis vectors for $\mathbb{R}^n$. Then the $m \times n$ matrix

$$A = (Te_1 \mid Te_2 \mid \ldots \mid Te_n)$$

has the property that

$$T(x) = Ax \text{ for all } x \in \mathbb{R}^n.$$  

E.g. In the example of the previous slide,

$$Te_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad Te_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

so the representing matrix is
A 5-point star with vertices $A(1, 5), B(4, 3), C(3, -1), D(-1, -1)$ and $E(-2, 3)$.
Example

Find and draw the image of the 5-point star under the linear map $T_M$ defined by the matrix $M = \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}$.

Solution
Solution (Continued)
Rotation about $0$ is linear

Consider the map $R_\alpha$, which rotates the $\mathbb{R}^2$ plane through an angle $\alpha$ anticlockwise about the origin.

One can show geometrically that $R_\alpha$ is a linear map see Section 7.3. example 3.

Example
Find the matrix $A$ representing $R_\alpha$.

Solution
Projection onto $b$ is linear

Recall given $b \in \mathbb{R}^n$ we have a projection map $\text{proj}_b : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which sends $x \mapsto \text{proj}_b x$.

**Proposition**

\[
\text{proj}_b x = \frac{1}{|b|^2} bb^T x
\]

Hence $\text{proj}_b$ is linear being the linear map associated to the matrix

\[
A = \frac{1}{|b|^2} bb^T.
\]

**Proof.**

Note

\[
Ax = \frac{1}{|b|^2} bb^T x = \frac{1}{|b|^2} b(b \cdot x) = \frac{b \cdot x}{|b|^2} b = \text{proj}_b x
\]

from the formula for $\text{proj}_b x$ given in MATH1131.
Let $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T = \text{proj}_b : \mathbb{R}^2 \to \mathbb{R}^2$.

i) Find the matrix $A$ representing $T$. ii) Check your answer by computing the linear map associated to the matrix $A$ you found.
Kernels of linear maps

Let \( T : V \to W \) be a linear map.

**Proposition-Definition**

The **kernel** of \( T \) (written \( \ker(T) \) or \( \ker T \)) is the set,

\[ \ker(T) = \{ v \in V \mid T(v) = 0 \} \subseteq V. \]

Let \( A \in M_{mn}(\mathbb{R}) \& \ T_A : \mathbb{R}^n \to \mathbb{R}^m \) be the assoc linear map. We define

\[ \ker A = \ker T_A = \{ v \in \mathbb{R}^n : Av = 0 \}. \]

\( \ker T \) is a subspace of \( V \).

**E.g.** Is \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) in \( \ker(2 - 1) \)?

**E.g.** Consider the differentiation map \( T : \mathbb{P} \to \mathbb{P}, (Tp)(x) = p'(x). \)
\( \ker T = \{ p \in \mathbb{P} \mid \frac{dp}{dx} = 0 \} = \mathbb{P}_0 \) the subspace of all constant polynomials.
Proof that kernels are subspaces

Let $T : V \rightarrow W$ be a linear map. We prove that $\ker T$ is a subspace of $V$ by checking closure axioms.

Proof.
Let $T : V \rightarrow W$ be a linear map.

**Proposition-Definition**

The **image** of $T$ is the set of all function values of $T$, that is,

$$\text{im}(T) = \{ w \in W : w = T(v) \text{ for some } v \in V \} \subseteq W.$$ 

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\text{im } A = \text{im } T_A = \{ b \in \mathbb{R}^m : b = Ax \text{ for some } x \in \mathbb{R}^n \} = \text{col}(A).$$

$\text{im } T$ is a subspace of $W$.

**Remark** Proof omitted, but note we already know $\text{col}(A)$ is a subspace as it is the span of the columns of $A$. 
Verifying whether or not vectors lie in the image

Example

Let \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \), \( \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \). Is \( \mathbf{b} \in \text{im } A \)?

Solution

The question amounts to asking: Can we write \( \mathbf{b} = A\mathbf{x} \) for some \( \mathbf{x} \in \mathbb{R}^2 \)? i.e. Can we solve \( A\mathbf{x} = \mathbf{b} \).
Example
Let \[
\begin{pmatrix}
1 & 2 & 3 & 1 \\
2 & 4 & 7 & 1 \\
1 & 2 & 2 & 2
\end{pmatrix}
\]. Find bases for \( \text{ker}(A) \) and \( \text{im}(A) = \text{col}(A) \).

Solution
\[
\begin{pmatrix}
1 & 2 & 3 & 1 \\
2 & 4 & 7 & 1 \\
1 & 2 & 2 & 2
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 2 & 3 & 1 \\
0 & 0 & 1 & -1 \\
0 & 0 & -1 & 1
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 2 & 3 & 1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}
= U
\]

The row echelon form \( U \) has first & third columns leading.
Solution (Continued)
Why study kernels & image?

$\text{im } T$ tells you about existence of solutions. $T \mathbf{x} = \mathbf{b}$ has a solution iff $\mathbf{b} \in \text{im } T$.

$\text{ker } T$ tells you about uniqueness of solutions. $T \mathbf{x} = \mathbf{0}$ has unique solution $\mathbf{x} = \mathbf{0}$ iff $\text{ker } T = \{0\}$. We’ll see later, that it also tells you about solutions to $T \mathbf{x} = \mathbf{b}$. 
Let $T : V \rightarrow W$ be a linear map & $A$ a matrix with associated linear map $T_A$.

**Definition**

- The **nullity** of $T$ is $\text{nullity}(T) = \dim \ker(T)$.
- The **nullity** of $A$ is $\text{nullity}(A) = \text{nullity}(T_A) = \dim \ker(A)$.
- The **rank** of $T$ is $\text{rank}(T) = \dim \text{im}(T)$.
- The **rank** of $A$ is $\text{rank}(A) = \text{rank}(T_A) = \dim \text{im}(A)$. 
Example (Continued from the example on p.30)

Let \( A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix} \). Find \( \text{nullity}(A) \) and \( \text{rank}(A) \).

Solution

A basis for \( \text{im} \ A \) was

Recall a basis for \( \text{ker} \ A \) had

Note that the basis vectors for \( \text{im} \ A \) corresponded to the leading columns of the row-echelon form \( U \) whilst the basis vectors for \( \text{ker} \ A \) corresponded to the non-leading columns.
The previous examples illustrates

**Key Lemma**

Let $A$ be an $m \times n$ matrix which reduces to a row-echelon form $U$.

1. $\text{nullity}(A) = \text{no. parameters in the general soln to } Ax = 0$
   $= \text{the number of non-leading columns of } U.$

2. $\text{rank}(A) = \text{the maximal no. independent columns of } A$
   $= \text{the number of leading columns of } U.$
Rank-Nullity Theorem

Our key lemma gives

**Theorem (Rank-nullity Theorem for Matrices)**

If $A$ is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$ 

**Proof.**

This can be used to prove more generally,

**Theorem (Rank-nullity Theorem for Linear Maps)**

Let $T : V \rightarrow W$ be a linear map with $V$ finite dimensional. Then

$$\text{rank}(T) + \text{nullity}(T) = \text{dim}(V).$$
Example of rank-nullity theorem

**Example**

Let \( \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the projection map \( \text{proj}_b \).

Verify the rank-nullity theorem in this case.

**Solution**
Nature of solutions to $Ax = b$ 

Our key lemma also gives

Theorem

The equation $Ax = b$ has:

1. no solution if $\text{rank}(A) \neq \text{rank}([A|b])$, and
2. at least one solution if $\text{rank}(A) = \text{rank}([A|b])$. Further,
   
   i) if $\text{nullity}(A) = 0$ the solution is unique, whereas,
   
   ii) if $\text{nullity}(A) = \nu > 0$, then the general solution is of the form

   $$x = x_p + \lambda_1 k_1 + \cdots + \lambda_\nu k_\nu \quad \text{for } \lambda_1, \ldots, \lambda_\nu \in \mathbb{R},$$

   where $x_p$ is any solution of $Ax = b$, and where $\{k_1, \ldots, k_\nu\}$ is a basis for $\ker(A)$. 
A theoretical application of rank-nullity theorem

Example

Prove that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then the following are equivalent.

a) For all $b \in \mathbb{R}^n$, there is at least one solution to $Tx = b$.

b) For all $b \in \mathbb{R}^n$, there is at most one solution to $Tx = b$.

Solution
In what sense are second order linear differential equations linear?

They involve the linear map \( T : C^2[\mathbb{R}] \rightarrow C[\mathbb{R}] \), where \( C^2[\mathbb{R}] \) is the vector space of all \( \mathbb{R} \)-valued functions with continuous second derivatives and \( C[\mathbb{R}] \) is the vector space of all continuous \( \mathbb{R} \)-valued functions—

\[
T(y) = a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy,
\]

where \( a, b, c \in \mathbb{R} \).

In this case, \( \text{ker}(T) \) is the solution set of the ODE —

\[
a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0,
\]

where \( a, b, c \in \mathbb{R} \).

Hence the homogeneous solution is a vector space. Furthermore, it is of dimension 2 i.e. \( \text{nullity}(T) = 2 \). We can also apply similar ideas for the solution to \( Ax = b \) to get the solution to

\[
a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x),
\]

where \( a, b, c \in \mathbb{R} \).
Example involving polynomials

To study boundary value problems, it’s useful to study linear maps such as the one below.

Example

The function \( T : \mathbb{P}_2 \rightarrow \mathbb{R}^2 \) is defined by \( Tp = \begin{pmatrix} p(1) \\ p'(1) \end{pmatrix} \)

a) Prove that \( T \) is linear.

b) Find \( \ker(T) \).

c) Use the rank-nullity theorem to find \( \text{im}(T) \).

Solution