MATH1231 Algebra, 2017
Chapter 7: Linear maps

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Chapter overview

- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f\binom{x}{y} = 2x - 3y$ is example of a linear function, $g\binom{x}{y} = x^2 - 5y$ is not.
- In this chapter, study more generally **linear transformations** $T : \mathbb{R}^m \to \mathbb{R}^n$.
- Even more gen, study linear $T : V \to W$ where $V, W =$ vector spaces $/\mathbb{F}$. Recall V is the **domain** of T & W the **codomain** of T.
- We'll generalise systems of linear equations $A\mathbf{x} = \mathbf{b}$ to "linear equations" of form $T\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \in W, \mathbf{x} \in V$.

Often abbrev $T(\mathbf{x}) = T\mathbf{x}$.

To define linear map, first consider

Addition Condition.

We say $T : V \longrightarrow W$ satisfies the **addition condition**, if

$$T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}') \text{ for all } \mathbf{v}, \mathbf{v}' \in V.$$

E.g. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T\binom{x}{y} = 2x - 3y$ satisfies the addn condn since given $\binom{x}{y}, \binom{x'}{y'} \in \mathbb{R}^2$

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix}\right) =$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} + T\begin{pmatrix} x' \\ y' \end{pmatrix} =$$

Warning The $+$ on the two sides of the equation are different!

Scalar Multiplication Condition

Scalar Multiplication Condition.

We say $T : V \longrightarrow W$ satisfies the **scalar multiplication condition**, if

$$T(\lambda \mathbf{v}) = \lambda T(\mathbf{v}) \text{ for all } \lambda \in \mathbb{F} \text{ and } \mathbf{v} \in V.$$

E.g. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T\binom{x}{y} = 2x - 3y$ satisfies the scalar multn condn since given $\binom{x}{y} \in \mathbb{R}^2, \lambda \in \mathbb{R}$

$$T\left(\lambda \begin{pmatrix} x \\ y \end{pmatrix}\right) =$$

$$\lambda T \begin{pmatrix} x \\ y \end{pmatrix} =$$

Warning The scalar multn on the two sides of the eqn are different!

Linear Transformation

Definition

Let $V, W =$ vector spaces $/\mathbb{F}$. A function $T : V \to W$ is called a **linear map** or a **linear transformation** if following both hold.

Addition Condition.

$$T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}') \text{ for all } \mathbf{v}, \mathbf{v}' \in V, \text{ and}$$

Scalar Multiplication Condition.

$$T(\lambda \mathbf{v}) = \lambda T(\mathbf{v}) \text{ for all } \lambda \in \mathbb{F} \text{ and } \mathbf{v} \in V.$$

E.g. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T\binom{x}{y} = 2x - 3y$ is linear.

Example

Show that the function $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = \begin{pmatrix} 4x_2 - 3x_3 \\ x_1 + 2x_2 \end{pmatrix} \quad \text{for} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$$

is a linear map.

Solution

Solution (Continued)

Solution (Continued)

Linear maps preserve zero.

Proposition.

If $T : V \longrightarrow W$ is a linear map, then $T(\mathbf{0}) = \mathbf{0}$.

Proof. $T(\mathbf{0}) = T(0\mathbf{0}) = 0T(\mathbf{0}) = \mathbf{0}$.

Example

Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 - 2 \\ x_1 \end{pmatrix}$ is not linear.

Soln

Another non-linear example

Example

Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2^2 \end{pmatrix}$$

is not linear.

Solution

Checking $T(\mathbf{0}) = \mathbf{0}$ here tells you nothing about linearity.
Suffice check that $T(\lambda \mathbf{v}) = \lambda T \mathbf{v}$ fails for single choice of pair $\lambda, \mathbf{v}$.

Alternate characterisation of linearity

We can combine the addn condn & scalar multn condn into one!

Theorem

The function $T : V \to W$ is a linear map iff for all $\lambda \in \mathbb{F}$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$

$$T(\lambda \mathbf{v}_1 + \mathbf{v}_2) = \lambda T(\mathbf{v}_1) + T(\mathbf{v}_2).$$

Remark This means that a linear map T has the special property that it sends the line $\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}$ to the line $\mathbf{x} = T\mathbf{a} + \lambda T\mathbf{v}$ or point $T\mathbf{a}$ if $T\mathbf{v} = \mathbf{0}$.
E.g. Differentation is a linear map. More precisely, we define $T : \mathbb{P} \longrightarrow \mathbb{P}$ by $Tp = \frac{dp}{dx}$. Then for $p, q \in \mathbb{P}, \lambda \in \mathbb{R}$

$$T(\lambda p + q) =$$

Theorem

If $T : V \to W$ *is linear map,* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ *&* $\lambda_1, \ldots, \lambda_n$ *are scalars, then*

$$T(\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n) = \lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_n T(\mathbf{v}_n).$$

Example

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ a function such that

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; T\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Show that T is not linear.

Solution

Solution (Continued)

Example

Given that T is a linear map and

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix},\ T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix},\ T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix}.$$

Find $T\begin{pmatrix}x\\y\\z\end{pmatrix}$.

Solution

Linear maps are determined by the values on a spanning set

Previous eg illustrates the following general result.

Theorem

Let $T : V \longrightarrow W$ be linear & $V = span(\mathbf{v}_1, \ldots, \mathbf{v}_m)$. Then T is completely determined by the m values $T\mathbf{v}_1, \ldots, T\mathbf{v}_m$.

Compare with the following:

An affine linear function $f(x) = mx + b$ is determined by two of its values $f(x_1), f(x_2)$, since its graph is a line which is determined by two points.

Matrices define Linear Maps

Theorem

For each $m \times n$ matrix A, the function $T_A : \mathbb{R}^n \to \mathbb{R}^m$, defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \qquad \textit{for} \quad \mathbf{x} \in \mathbb{R}^n,$$

is a linear map called the **associated linear map***.*

Proof.

□

Example of reflection

Example

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, describe the associated linear map T_A geometrically as a mapping $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Solution

Matrix Representation Theorem

Conversely, given linear $T : \mathbb{R}^n \to \mathbb{R}^m$, we can find an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. In this case, we say A is a matrix **representing** T.

Example

Given that $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x - y \\ y \end{pmatrix}$ is linear. Find the matrix A representing T.

Solution

Formula for representing matrix

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let the vectors $\mathbf{e}_j$ for $1 \leqslant j \leqslant n$ be the standard basis vectors for $\mathbb{R}^n$. Then the $m \times n$ matrix

$$A = (T\mathbf{e}_1 | T\mathbf{e}_2 | \dots | T\mathbf{e}_n)$$

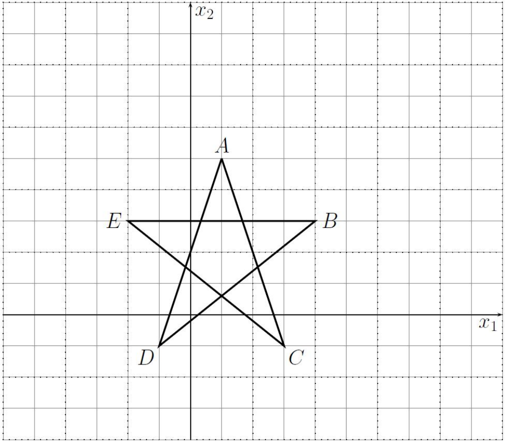
has the property that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

E.g. In the example of the previous slide,

$T\mathbf{e}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $T\mathbf{e}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ so the representing matrix is

Stretching and Compressing



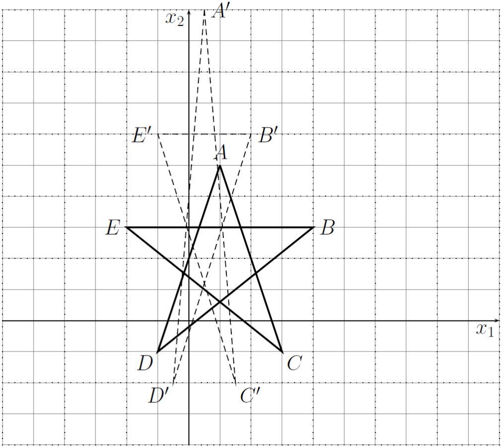
A 5-point star with vertices
$A(1,5)$, $B(4,3)$, $C(3,-1)$, $D(-1,-1)$ and $E(-2,3)$.

Example

Find and draw the image of the 5-point star under the linear map T_M defined by the matrix $M = \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}$.

Solution

Solution (Continued)



Rotation about $\mathbf{0}$ is linear

Consider the map R_α, which rotates the $\mathbb{R}^2$ plane through an angle α anticlockwise about the origin.

One can show geometrically that R_α is a linear map see Section 7.3. example 3.

Example

Find the matrix A representing R_α.

Solution

Projection onto $\mathbf{b}$ is linear

Recall given given $\mathbf{b} \in \mathbb{R}^n$ we have a projection map $\text{proj}_{\mathbf{b}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which sends $\mathbf{x} \mapsto \text{proj}_{\mathbf{b}}\mathbf{x}$.

Proposition

$$\text{proj}_{\mathbf{b}}\mathbf{x} = \frac{1}{|\mathbf{b}|^2}\mathbf{b}\mathbf{b}^T\mathbf{x}$$

Hence $\text{proj}_{\mathbf{b}}$ is linear being the linear map associated to the matrix

$$A = \frac{1}{|\mathbf{b}|^2}\mathbf{b}\mathbf{b}^T.$$

Proof.

Note

$$A\mathbf{x} = \frac{1}{|\mathbf{b}|^2}\mathbf{b}\mathbf{b}^T\mathbf{x} = \frac{1}{|\mathbf{b}|^2}\mathbf{b}(\mathbf{b}\cdot\mathbf{x}) = \frac{\mathbf{b}\cdot\mathbf{x}}{|\mathbf{b}|^2}\mathbf{b} = \text{proj}_{\mathbf{b}}\mathbf{x}$$

from the formula for $\text{proj}_{\mathbf{b}}\mathbf{x}$ given in MATH1131 □

Sample projection

Example

Let $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T = \text{proj}_{\mathbf{b}} : \mathbb{R}^2 \to \mathbb{R}^2$.

i) Find the matrix A representing T. ii) Check your answer by computing the linear map associated to the matrix A you found.

Solution

Kernels of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **kernel** of T (written $\ker(T)$ or $\ker T$) is the set,

$$\ker(T) = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}\} \subseteq V.$$

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\ker A = \ker T_A = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\}.$$

$\ker T$ is a subspace of V.

E.g. Is $\binom{1}{2}$ in $\ker(2 \ \ -1)$?

E.g. Consider the differentiation map $T : \mathbb{P} \longrightarrow \mathbb{P}, (Tp)(x) = p'(x)$.
$\ker T = \{p \in \mathbb{P} | \frac{dp}{dx} = 0\} = \mathbb{P}_0$ the subspace of all constant polynomials.

Proof that kernels are subspaces

Let $T : V \longrightarrow W$ be a linear map. We prove that $\ker T$ is a subspace of V by checking closure axioms.

Proof.

$\square$

Images of linear maps

Let $T : V \to W$ be a linear map.

Proposition-Definition

The **image** of T is the set of all function values of T, that is,

$$\text{im}(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \quad \text{for some} \quad \mathbf{v} \in V\} \subseteq W.$$

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\text{im } A = \text{im } T_A = \{\mathbf{b} \in \mathbb{R}^m : \mathbf{b} = A\mathbf{x} \quad \text{for some} \quad \mathbf{x} \in \mathbb{R}^n\} = \text{col}(A).$$

$\text{im}\, T$ is a subspace of W.

Remark Proof omitted, but note we already know $\text{col}(A)$ is a subspace as it is the span of the columns of A.

Verifying whether or not vectors lie in the image

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$. Is $\mathbf{b} \in \text{im } A$?

Solution

The question amounts to asking: Can we write $\mathbf{b} = A\mathbf{x}$ *for some* $\mathbf{x} \in \mathbb{R}^2$*?*
i.e. Can we solve $A\mathbf{x} = \mathbf{b}$.

Finding Bases for Kernels & Images.

Example

Let $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix}$. Find bases for $\ker(A)$ and $\text{im}(A) = \text{col}(A)$.

Solution

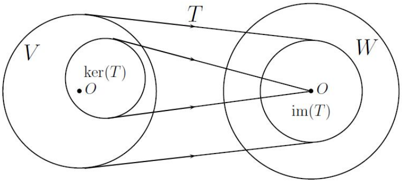
$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

The row echelon form U has first & third columns leading.

Solution (Continued)

Solution (Continued)

Why study kernels & image?



im T tells you about existence of solutions. $T\mathbf{x} = \mathbf{b}$ has a solution iff $\mathbf{b} \in \operatorname{im} T$.

ker T tells you about uniqueness of solutions. $T\mathbf{x} = \mathbf{0}$ has unique solution $\mathbf{x} = \mathbf{0}$ iff $\ker T = \mathbf{0}$. We'll see later, that it also tells you about solutions to $T\mathbf{x} = \mathbf{b}$.

Rank and Nullity

Let $T : V \longrightarrow W$ be a linear map & A a matrix with associated linear map T_A.

Definition

- The **nullity** of T is $\text{nullity}(T) = \dim \ker(T)$.
- The **nullity** of A is $\text{nullity}(A) = \text{nullity}(T_A) = \dim \ker(A)$.
- The **rank** of T is $\text{rank}(T) = \dim \text{im}(T)$.
- The **rank** of A is $\text{rank}(A) = \text{rank}(T_A) = \dim \text{im}(A)$.

Rank, Nullity example

Example (Continued from the example on p.30)

Let $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix}$. Find nullity($A$) and rank($A$).

Solution

A basis for im A was

Recall a basis for ker A had

Note that the basis vectors for im A corresponded to the leading columns of the row-echelon form U whilst the basis vectors for ker A corresponded to the non-leading columns.

Rank & Nullity from the row-echelon form

The previous examples illustrates

Key Lemma

Let A be an $m \times n$ matrix which reduces to a row-echelon form U.

1. $\text{nullity}(A) =$ no. parameters in the general soln to $A\mathbf{x} = \mathbf{0}$
   $=$ the number of non-leading columns of U.
2. $\text{rank}(A) =$ the maximal no. independent columns of A
   $=$ the number of leading columns of U.

Rank-Nullity Theorem

Our key lemma gives

Theorem (Rank-nullity Theorem for Matrices)

If A is an $m \times n$ matrix, then

$$rank(A) + nullity(A) = n.$$

Proof.

□

This can be used to prove more generally,

Theorem (Rank-nullity Theorem for Linear Maps)

Let $T : V \to W$ be a linear map with V finite dimensional. Then

$$rank(T) + nullity(T) = dim(V).$$

Example of rank-nullity theorem

Example

Let $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection map $\text{proj}_{\mathbf{b}}$.
Verify the rank-nullity theorem in this case.

Solution

Nature of solutions to $A\mathbf{x} = \mathbf{b}$

Our key lemma also gives

Theorem

The equation $A\mathbf{x} = \mathbf{b}$ has:

1. *no solution if rank$(A) \neq$ rank$([A|\mathbf{b}])$, and*
2. *at least one solution if rank$(A) =$ rank$([A|\mathbf{b}])$. Further,*
   - i) *if nullity$(A) = 0$ the solution is unique, whereas,*
   - ii) *if nullity$(A) = \nu > 0$, then the general solution is of the form*

$$\mathbf{x} = \mathbf{x}_p + \lambda_1 \mathbf{k}_1 + \cdots + \lambda_\nu \mathbf{k}_\nu \quad \text{for } \lambda_1, \ldots, \lambda_\nu \in \mathbb{R},$$

where $\mathbf{x}_p$ is any solution of $A\mathbf{x} = \mathbf{b}$, and where $\{\mathbf{k}_1, \ldots, \mathbf{k}_\nu\}$ is a basis for ker(A).

A theoretical application of rank-nullity theorem

Example

Prove that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then the following are equivalent.

a) For all $\mathbf{b} \in \mathbb{R}^n$, there is at least one solution to $T\mathbf{x} = \mathbf{b}$.

b) For all $\mathbf{b} \in \mathbb{R}^n$, there is at most one solution to $T\mathbf{x} = \mathbf{b}$

Solution

Q In what sense are second order linear differential equations linear?
A They involve the linear map $T : \mathcal{C}^2[\mathbb{R}] \to \mathcal{C}[\mathbb{R}]$, where $\mathcal{C}^2[\mathbb{R}]$ is the vector space of all $\mathbb{R}$-valued functions with continuous second derivatives and $\mathcal{C}[\mathbb{R}]$ is the vector space of all continuous $\mathbb{R}$-valued functions—

$$T(y) = a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy, \quad \text{where } a, b, c \in \mathbb{R}.$$

In this case, $\ker(T)$ is the solution set of the ODE —

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = 0, \quad \text{where } a, b, c \in \mathbb{R}.$$

Hence the homogeneous solution is a vector space. Furthermore, it is of dimension 2 i.e. $\text{nullity}(T) = 2$. We can also apply similar ideas for the solution to $A\mathbf{x} = \mathbf{b}$ to get the solution to

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = f(x), \quad \text{where } a, b, c \in \mathbb{R}.$$

Example involving polynomials

To study boundary value problems, it's useful to study linear maps such as the one below.

Example

The function $T : \mathbb{P}_2 \longrightarrow \mathbb{R}^2$ is defined by $Tp = \binom{p(1)}{p'(1)}$

a) Prove that T is linear.

b) Find $\ker(T)$.

c) Use the rank-nullity theorem to find $\operatorname{im}(T)$.

Solution

Solution (Continued)