MATH1231 Algebra, 2017
Chapter 6

A/Prof. Daniel Chan

School of Mathematics and Statistics
University of New South Wales

danielc@unsw.edu.au

Things to keep in mind before we start

1. Go to Moodle 1231 page often.
2. Do the revision questions in the "yellow" notes.
3. Lecture notes are uploaded chapter by chapter.
4. Print the lecture notes and bring them to lectures.
5. Lecture recordings will be available.
6. In this course, there are lots of new concepts and definitions. There are fewer computations but success requires knowing which computations to perform and arguments to express.

Review of parametric form for lines

Recall the parametric form for a line in space passing through **a** in direction **v**:

$$\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}, \quad \lambda \in \mathbb{R}.$$

Similarly, given $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$, the parametric form above defines a line in $\mathbb{R}^n$.
E.g. Let's solve the DE $\frac{dy}{dx} = 2x$.

Question

Is the set of solutions

$$y = x^2 + \lambda = x^2 + \lambda 1, \quad \lambda \in \mathbb{R}$$

a "line" of polynomials?

Answer YES, as long as you properly define an environment in which you can talk about lines & planes in an abstract general setting!
The goal of this chapter is to define precisely such an environment.

Informal “definition” of vector spaces

A **vector space** is an environment in which you can talk about linear concepts such as lines.

Q What does this require?

At a minimum you need the following 4 ingredients:

V — a set of objects called **vectors** e.g. $\mathbb{R}^n$ or arrows,

$\mathbb{F}$ — a set of scalars (numbers), for us either $\mathbb{R}$ or $\mathbb{C}$,

$+$ — an operation for adding two (possibly equal) elements $\mathbf{u}, \mathbf{v}$ in V to obtain another $\mathbf{u} + \mathbf{v} \in V$, and

$*$ — an operation for multiplying an element $\mathbf{v} \in V$ by a scalar $\lambda \in \mathbb{F}$, to obtain the vector $\lambda * \mathbf{v} \in V$ which is usually abbreviated to $\lambda\mathbf{v}$.

Given a “vector system” $(V, +, *, \mathbb{F})$, we also want the addn & scalar multn to behave sensibly e.g. $2(2\mathbf{v}) = 4\mathbf{v}$.

Q What’s sensible mean here?

The system has to satisfy the 8 **axioms** on the next few slides.

Definition of a Vector Space

Definition (Vector Space).

A **vector space** V over the set of scalars $\mathbb{F}$ is a vector system $(V, +, *, \mathbb{F})$ which obeys the following 8 axioms.

1. **Associative law of addition**.
   If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ then $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
2. **Commutative law of addition**.
   If $\mathbf{u}, \mathbf{v} \in V$ then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. **Existence of zero**. There is a special element $\mathbf{0}$ in V called the **zero vector** which has the property that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
4. **Existence of Negative**. For each $\mathbf{v} \in V$ there exists an element $\mathbf{w} \in V$ (the **negative** of $\mathbf{v}$, i.e. $\mathbf{w} = -\mathbf{v}$) such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.

and

Vector space axioms cont'd

5. **Associative law of multiplication by a scalar**. If $\lambda, \mu \in \mathbb{F}$ and $\mathbf{v} \in V$ then $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$.
6. If $\mathbf{v} \in V$ then $1\mathbf{v} = \mathbf{v}$.
7. **Scalar distributive law**. If $\lambda, \mu \in \mathbb{F}$ and $\mathbf{v} \in V$ then $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$.
8. **Vector distributive law**. If $\lambda \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in V$ then $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$.

Remark

We usually abbreviate $(V, +, *, \mathbb{F})$ to V which is fine if $+, *, \mathbb{F}$ are understood.

What are axioms anyway?

Question

What's a mammal?

A It's a type of animal which satisfies the following axiom(s)

Warning I didn't define what an animal was! But I also didn't define what a set was!!

Axioms are important because they

- define classes of objects, and
- give the starting assumptions for proofs.

The vector space $\mathbb{R}^n$

The co-ordinatewise addition and scalar multiplication of $\mathbb{R}^n$ defined in 1131 are called the usual addition and the usual scalar multiplication.

For any $\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}, \quad \text{and} \quad \lambda \mathbf{a} = \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix}.$$

Theorem

The $V = \mathbb{R}^n$ with the usual addn & scalar multn is a vector space over $\mathbb{R}$.

Hopefully, you saw all the axioms verified in MATH1131.

Droids

Vector systems can get pretty crazy. Consider following vector system $(V, +, *, \mathbb{R})$.
V = the set of droids {C-3PO, R2-D2, BB-8}.
$+$ — the sum of any two droids = BB-8
$*$ — the scalar multiple of any droid = R2-D2
Q What's 5 R2-D2 + C-3PO?

Example

Prove that V satisfies the commutative law of addition!

In fact, V also satisfies the associative laws of addn & scalar multn!

Proving vector systems are not vector spaces

Example

Prove that the vector system of droids is not a vector space.

Proof.

To show a vector system is not a vector space, you just need to show it fails one of the axioms.
Think of the axioms as like a checklist — pass all & you are in good shape, fail one & your out.

Review polynomials

Recall that a polynomial over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ of degree k is a function $p : \mathbb{F} \to \mathbb{F}$ such that

$$p(x) = a_0 + a_1 x + \cdots + a_k x^k, \text{ where } a_0, a_1, \ldots, a_k \in \mathbb{F} \text{ and } a_k \neq 0.$$

The zero polynomial defined by $p(x) = 0$ has degree $-\infty$ by defn. Let $\mathbb{P}(\mathbb{F})$ = set of polynomials over $\mathbb{F}$. Let $p, q \in \mathbb{P}$ & $\lambda \in \mathbb{F}$. Define polynomials $p + q$ and λp by

$$(p+q)(x) = p(x) + q(x) \quad \text{and} \quad (\lambda p)(x) = \lambda p(x),$$

for all $x \in \mathbb{F}$.

NB These operations correspond to summing corresponding co-efficients and scalar multiplying them.

Eg

The vector space $\mathbb{P}$

Oft abbreviate $\mathbb{P}(\mathbb{F})$ to $\mathbb{P}$ if $\mathbb{F}$ understood.

Theorem

The set $\mathbb{P}$ with addn & scalar multn defined above is a vector space $/\mathbb{F}$.

Proof. Just check all 8 axioms.

Remark This means we can talk about linear concepts in $\mathbb{P}$ e.g. $x^2 + \lambda, \lambda \in \mathbb{R}$ is a line of polynomials.

On the other hand, we can't speak of lines & planes of droids. Why?

Axioms allow usual vector arithmetic

The vector space axioms allow you to perform the usual algebraic manipulations of vector arithmetic.

Example

Let V be a vector space $/\mathbb{R}$ & $\mathbf{u}, \mathbf{v} \in V$. Simplify $3(\mathbf{u} + \mathbf{v}) + \mathbf{v}$ stating all axioms you use.

Other important examples of vector spaces

Examples

The following are also examples of vector spaces.

- The vector space of all 2-dim geometric vectors over $\mathbb{R}$ with usual vector addn & scalar multn
- The vector space of all 3-dim geometric vectors over $\mathbb{R}$ with usual vector addn & scalar multn
- $\mathbb{C}^n$ over $\mathbb{C}$.
  Addn & scalar multn are co-ordinatewise.
- The vector space of all $m \times n$ matrices, $M_{mn}(\mathbb{F})$ over $\mathbb{F}$.
  with matrix addn & scalar multn.
- The set of all real-valued functions with domain X, $\mathcal{R}[X]$ over $\mathbb{R}$.
  Addn & scalar multn are pointwise.

Vector subtraction

Proposition 1.

In any vector space V, the following properties hold.

1. **Uniqueness of Zero.** *There is one and only one zero vector.*
2. **Cancellation Property.** *If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ satisfy*
   $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, *then* $\mathbf{v} = \mathbf{w}$.
3. **Uniqueness of Negatives.** *For all $\mathbf{v} \in V$, there exists only one $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$. We write $-\mathbf{v} = \mathbf{w}$*

Remark Negatives allow us to define vector subtraction

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

Vector arithmetic

The axioms imply all sorts of other rules of vector arithmetic hold.

Proposition 2.

Suppose that $\mathbf{v}$ *is a vector in a vector space* V*;* λ *is a scalar;* 0 *is the zero scalar;* $\mathbf{0}$ *is the zero vector in* V*. Then the following properties hold.*

1. $\lambda\mathbf{0} = \mathbf{0}$.
2. $0\mathbf{v} = \mathbf{0}$.
3. $(-1)\mathbf{v} = -\mathbf{v}$. *Here* -1 *is a scalar and* $-\mathbf{v}$ *is the additive inverse of* $\mathbf{v}$.
4. *If* $\lambda\mathbf{v} = \mathbf{0}$*, then either* $\lambda = 0$ *or* $\mathbf{v} = \mathbf{0}$.
5. *If* $\lambda\mathbf{v} = \mu\mathbf{v}$ *and* $\mathbf{v} \neq \mathbf{0}$ *then* $\lambda = \mu$.

Sample proof

Example

Prove Proposition 2.3 assuming Proposition 2.2.

Proof.

□

Let V be a vector space $/\mathbb{F}$.

Definition

We say a subset $S \subseteq V$ is **closed under addition** if for every $\mathbf{u}, \mathbf{v} \in S$ we have $\mathbf{u} + \mathbf{v} \in S$.

E.g. Let V be the vector space of 2-dim geometric vectors. Let S be the subset of vectors parallel to the x or y axis. DRAW PICTURE.

so S is NOT closed under addition.

Closure under scalar multiplication

Definition

We say a subset $S \subseteq V$ is **closed under scalar multiplication** if for every $\mathbf{v} \in S$ & scalar λ we have $\lambda\mathbf{v} \in S$.

E.g. Let V be the vector space of 2-dim geometric vectors. Let S be the subset of vectors parallel to the x or y axis.

so S is closed under scalar multiplication.

Subspaces

An easy way to come up with more examples of vector spaces is as follows.

Theorem-Definition

A subset S of a vector space $V/\mathbb{F}$ is a **subspace of** V if

i) the zero vector of V belongs to S;

ii) S is closed under addn; and

iii) S is closed under scalar multn.

In this case, S is itself a vector space $/\mathbb{F}$ with addn & scalar multn the "same" as that in V.

Why? The axioms are all inherited from V!

In the "yellow" notes, the above result is referred to as the Subspace Theorem.

Remark i), ii), iii) are sometimes called closure axioms for a subspace.

E.g. The subset $\mathbf{0} = \{\mathbf{0}\}$ is a subspace called the **zero subspace**.

Examples of subsets which are not subspaces

To check if a subset is a subspace, just go down the "checklist" of closure axioms.

Example

Is $S = \left\{ \mathbf{x} \in \mathbb{R}^3 : x_1 - 2x_2 + 3x_3 = 1 \right\}$ is a subspace of $\mathbb{R}^3$?

Solution

Example of a subspace

Example

Is $S = \{\mathbf{x} \in \mathbb{R}^3 : x_1 - 2x_2 + 3x_3 = 0\}$ a subspace of $\mathbb{R}^3$.

Solution

Yet another $\mathbb{R}^n$ example

Example

Is $S = \left\{\mathbf{x} \in \mathbb{R}^3 : x_3 = x_1^2 + x_2^2\right\}$ is a subspace of $\mathbb{R}^3$.

Solution

The subspace $\mathbb{P}_n$

Let $\mathbb{P}_n$ = set of polynomials of degree n or less.

Proposition

$\mathbb{P}_n$ is a subspace of $\mathbb{P}$.

Proof.

Consider 2 polynomials in $\mathbb{P}_n$ which must have form

$$p(x) = p_0 + p_1 x + \ldots + p_n x^n \quad \& \quad q(x) = q_0 + q_1 x + \ldots + q_n x^n$$

□

An Example of a subspace of $\mathbb{P}_n$.

Example

Show that $S = \{p \in \mathbb{P}_2(\mathbb{R}) : xp'(x) - 2p(x) = 0\}$ is a subspace of $\mathbb{P}_2$.

Solution

Solution (Continued)

A subspace of a subspace is a subspace

Proposition

If U is a subspace of V and V a subspace of W, then U is a subspace of W.

E.g. $S = \{p \in \mathbb{P}_2(\mathbb{R}) : xp'(x) - 2p(x) = 0\}$ is a subspace of $\mathbb{P}_2$ & hence also a subspace of $\mathbb{P}$.

Linear Combinations

Definition (Linear Combination).

Let V = vector space $/\mathbb{F}$ & $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$. Then a **linear combination** of S is a vector or expression of the form

$$\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n \quad \text{for some} \quad \lambda_1, \ldots, \lambda_n \in \mathbb{F}.$$

Proposition

Every linear combination of S is also a vector in V.

Why?

Examples of linear combinations

Suppose that $S = \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^3$. Here are some examples of linear combinations of S.

$$2 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$2 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

Span

Definition (Span)

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset$ vector space V. The **span** of S is the set of all linear combinations of S. It's denoted by $\text{span}(S)$ or $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

If $\text{span}(S) = V$, we say S is **spanning set** of V, or S **spans** V. By default, define $\text{span}(\varnothing) = \mathbf{0}$.

Example

Prove that $\{1,\ x,\ x^2\} \subset \mathbb{P}_2$ is a spanning set for $\mathbb{P}_2$.

Proof.

We know $\text{span}(1, x, x^2) \subseteq \mathbb{P}_2$ so suffice show $\mathbb{P}_2 \subseteq \text{span}(S)$. Any $p \in \mathbb{P}_2$ has form

$$p = a + bx + cx^2 = a(1) + b(x) + c(x^2) \text{ for some scalars } a, b, c.$$

$$\therefore \quad \text{span}(\{1,\ x,\ x^2\}) = \mathbb{P}_2 \ \&\ \{1,\ x,\ x^2\} \text{ is a spanning set for } \mathbb{P}_2. \qquad \square$$

Span & parametric forms for lines & planes

Eg. 1 Span$\left(\begin{pmatrix}1\\1\end{pmatrix}\right)$ = set of all $\lambda\begin{pmatrix}1\\1\end{pmatrix}, \lambda \in \mathbb{R}$.

Eg. 2 Span$\left(\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\4\\1\end{pmatrix}\right) =$

Determining if a vector lies in a span?

Example

Let $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \in \mathbb{R}^3$, and the set $S = \left\{ \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$. Is $\mathbf{u} \in \text{span}(S)$? If so write $\mathbf{u}$ as a linear combination of S. How about $\mathbf{v}$?

Solution

Solution (Continued)

Solution (Continued)

Example (Continue from the previous example.)

Is S a spanning set for $\mathbb{R}^3$? If not, find conditions on $\mathbf{b} \in \mathbb{R}^3$ to be in the span(S). Give a geometric interpretation of span(S).

Solution

Solution (Continued)

An example in $\mathbb{P}_n$

Example

Is the set

$$S = \{1 + 2x + 3x^2,\ 2 + 4x + x^2,\ 1 + 2x + 8x^2,\ 1 - x + 4x^2\}$$

a spanning set for $\mathbb{P}_2$?

Solution

Solution (Continued)

Solution (Continued)

Rem A subset spans $\iff$ associated linear eqns always has a soln.
Existence of solns $\longleftrightarrow$ spanning.

Properties of a span

Theorem

Let S = finite subset of vector space V. Then span(S) is a subspace of V.

Proof.

Span$(\varnothing) = \mathbf{0}$ *is a subspace so we may assume* $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

- *Since* $\mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n$, *the zero vector of* V *is a linear combination of* S *i.e.* $\mathbf{0} \in$ *Span*(S).
- *Check Span*(S) *closed under addition.*

  Any $\mathbf{u}, \mathbf{v} \in$ *Span*(S), *have form*

$$\begin{aligned} \mathbf{u} &= \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n, \quad \lambda_1, \ldots, \lambda_n \text{ are scalars} \\ \mathbf{v} &= \mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n, \quad \mu_1, \ldots, \mu_n \text{ are scalars.} \end{aligned}$$

Solution (Continued)

Hence, by the commutative law, associative law of addition, and the scalar distributive law

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= (\lambda_1\mathbf{v}_1+\cdots+\lambda_n\mathbf{v}_n)+(\mu_1\mathbf{v}_1+\cdots+\mu_n\mathbf{v}_n) \\ &= (\lambda_1+\mu_1)\mathbf{v}_1+\cdots+(\lambda_n+\mu_n)\mathbf{v}_n \end{aligned}$$

$\therefore$ $\mathbf{u}+\mathbf{v}$ *is a linear combination of* S *and* $\mathbf{u}+\mathbf{v} \in$ *Span*(S).

Span(S) *is closed under addition.*

- *Check Span*(S) *closed under scalar multn.*

  For any scalar λ *&* $\mathbf{u} \in$ *Span*(S) *as above we have ,*

$$\begin{aligned} \lambda\mathbf{u} &= \lambda(\lambda_1\mathbf{v}_1+\cdots+\lambda_n\mathbf{v}_n) \\ &= \lambda(\lambda_1\mathbf{v}_1)+\cdots+\lambda(\lambda_n\mathbf{v}_n) \quad \text{[Vector distributive law]} \\ &= (\lambda\lambda_1)\mathbf{v}_1+\cdots+(\lambda\lambda_n)\mathbf{v}_n \quad \text{[Scalar assocative law]} \end{aligned}$$

Proof (Continued).

Hence $\lambda\mathbf{u}$ is a linear combination of S and $\lambda\mathbf{u} \in Span(S)$.

$Span(S)$ is closed under scalar multiplication.

By the Subspace Thm-Defn, span(S) is a subspace of V.

□

Remark

Any subspace of V containing a finite set of vectors S contains span(S). Hence, span(S) is the smallest subspace containing S.

Subspaces of $\mathbb{R}^3$

Proposition

The only subspaces of $\mathbb{R}^3$ are $\{\mathbf{0}\}$, lines and planes through the origin and $\mathbb{R}^3$ itself.

Why? Let $W \subseteq \mathbb{R}^3$ be a subspace of $\mathbb{R}^3$. If $W \neq \mathbf{0}$ then pick non-zero $\mathbf{u} \in W$.

Then W contains the line span($\mathbf{u}$).

Either $W =$ the line span($\mathbf{u}$) or we can pick $\mathbf{v}$ in $W -$ span($\mathbf{u}$), so W contains the plane span($\mathbf{u}, \mathbf{v}$).

Relationship between matrices and spans in $\mathbb{R}^m$

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^m$. Let $A = \text{matrix } (\mathbf{v}_1|\cdots|\mathbf{v}_n)$. Then $\mathbf{b} \in \text{span}(S) \iff$ there exist scalars $x_1, \ldots, x_n$ such that

$$\mathbf{b} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = (\mathbf{v}_1|\cdots|\mathbf{v}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Upshot: $\mathbf{b}$ is a linear combination of $S \iff \mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$.

In other words, $\mathbf{b} \in \text{span}(S) \iff A\mathbf{x} = \mathbf{b}$ has a soln $\mathbf{x} \in \mathbb{R}^n$. [Algebra Notes: Proposition 3 in 6.4]

This motivates the next slide.

Column Space

Definition

The subspace of $\mathbb{R}^m$ spanned by the columns of an $m \times n$-matrix A is called the **column space** of A and is denoted by $\text{col}(A)$.

Example

Is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in the column space of $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{pmatrix}$.

Solution

Solution (Continued)

Linear Independence

Definition

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset$ vector space V. We say S is **linearly independent** if

$$(*) \qquad \mathbf{0} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n$$

implies $\lambda_1 = \cdots = \lambda_n = 0$.
Otherwise say S is **linearly dependent** i.e. (*) above has a soln for scalars $\lambda_1, \dots, \lambda_n$ not all 0.

Eg Show $S = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\}$ is linearly dependent.

Next slide see linear dependence generalises of the notion of parallel vectors.

Linear dependence as a generalisation of parallel vectors

Theorem

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset$ vector space V. The following statements are equivalent.

1. S is linearly independent.
2. If $\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n = \mu_1 \mathbf{v}_1 + \cdots + \mu_n \mathbf{v}_n$ then $\lambda_1 = \mu_1, \dots, \lambda_n = \mu_n$.
3. None of the vectors in S is a linear combn of the other vectors in S.

Example

Two geometric vectors $\mathbf{u}, \mathbf{v}$ or vectors in $\mathbb{R}^n$ are linearly dependent iff they are parallel.

Why?

Checking Linear Independence

Example

Show that $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is a linearly independent set.

Thm then $\implies$ there is only one way to write $\mathbf{b} \in \text{span}(S)$ as a linear combination of S.

Solution

We solve

$$\mathbf{0} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Solution (Continued)

How does Span(S) change with S

Eg Find Span$\begin{pmatrix}1\\1\end{pmatrix}$), Span$\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$, Span$\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right)$,

Theorem

Let $S =$ finite subset of a vector space V. For any $\mathbf{v} \in V$, we have span$(S \cup \{\mathbf{v}\}) =$ span(S) if and only if $\mathbf{v} \in$ span(S).

What you can do with a linear dependence relation

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset$ vector space $V/\mathbb{F}$. If $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F}$ **not all zero** satisfy

$$(*) \qquad \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n = \mathbf{0},$$

then S is **linearly dependent set** & (*) is a called a **linear dependence relation**. Can find $\lambda_i \neq 0$.Hence,

$$\mathbf{v}_i = -\frac{\lambda_1}{\lambda_i}\mathbf{v}_1 - \cdots + \frac{\lambda_{i-1}}{\lambda_i}\mathbf{v}_{i-1} - \frac{\lambda_{i+1}}{\lambda_i}\mathbf{v}_{i+1} \cdots - \frac{\lambda_n}{\lambda_i}\mathbf{v}_n.$$

$\therefore$ $\mathbf{v}_i$ is a linear combn of $\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n$.
Thm last slide $\implies$

Upshot

In this case,

$$\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n).$$

Example

Suppose that

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}.$$

a) Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

b) Find a proper subset of S which has the same span as S.

Solution

Solution (Continued)

Important theorems regarding linear independence & span

Theorem

1. The span of every proper subset of S is a proper subspace of span(S) if and only if S is linearly independent.
2. If S is linearly independent and $\mathbf{v} \in V$ but not in span(S), then $S \cup \{\mathbf{v}\}$ is linearly independent.

E.g. Some examples with 3-dim geometric vectors.

Verifying linear (in)dependence in $\mathbb{P}_n$

Example

Is $S = \{1 + 2x - x^2,\ -3 - x - 2x^2,\ 2 + 3x + x^2\}$ a linearly independent subset of $\mathbb{P}_2$.

Proof.

Proof (Continued).

□

Why do we care about linearly independent spanning sets?

Question

Let $V =$ vector space of 2-dim or 3-dim geometric vectors. How do you construct a co-ordinate system on V.

A You need co-ordinate axes & scale on axes & $+$ve direction for axes
Equivalently, just give set S of unit positive position vectors for each axis.
But when is such a choice sensible?

Upshot S gives a sensible co-ordinate system iff S is a linearly independent spanning set. In this case, no. of elements of $S =$ no. co-ordinate axes.

Definition (Basis).

Let $V =$ vector space. A subset $B \subset V$ is a **basis for** V if a) B is linearly independent & b) $V = \text{span}(B)$.

E.g. Let $V =$ space of 2-dim geometric vectors and $S = \{\mathbf{u}, \mathbf{v}\}$ be any 2 non-parallel vectors (so are linearly independent).

Hence S is a basis for V. It is not unique!

Note special case The basis for the zero vector space $\mathbf{0}$ is $\varnothing$.

Standard basis for $\mathbb{R}^n$

Let $\mathbf{e}_i$ be the vector in $\mathbb{R}^n$ with the i-th entry 1 and all other entries 0. The set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a linearly independent spanning set of $\mathbb{R}^n$. Why?

$$(x_1 \quad x_2 \quad \cdots \quad x_n)^T = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

Hence, it is a basis which is called the **standard basis** of $\mathbb{R}^n$.

Example

In $\mathbb{R}^2$, the standard basis is $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

In $\mathbb{R}^3$, the standard basis is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

This basis is furthermore **orthonormal** in the sense that all basis vectors have length 1 i.e. $|\mathbf{e}_i| = 1$ and vectors are mutually orthogonal i.e. $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for $i \neq j$.

Standard basis for $\mathbb{P}_n$

Example

The set $\{1,\ x, \ldots,\ x^n\}$ is a basis for $\mathbb{P}_n$ called the **standard basis** for $\mathbb{P}_n$.

Why? Every polynomial in $\mathbb{P}_n$ can be written uniquely in the form

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n.$$

Verifying a Subset is a Basis

Example

Is $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\}$ a basis for $\mathbb{R}^3$?

Solution

Note that verifying spanning set and linear independence involve the same Gaussian elimination so only do it once!

Solution (Continued)

Example of a basis for $\mathbb{P}_n$

Example

Is

$$S = \{p_1(x) = 1 + 2x + x^2,\ p_2(x) = 1 + 3x + 2x^2,\ p_3(x) = -1 + 2x + 5x^2\}$$

a basis for $\mathbb{P}_2$? What about $\mathbb{P}_3$?

Solution

Solution (Continued)

Spanning sets are "bigger" than linearly independent sets

Theorem

Suppose a vector space V is spanned by a set S of s vectors. Then any linearly independent set in V has $\leqslant s$ vectors.

Proof. is hard and in notes. Result is believable e.g. $V =$ the space of 2-dim geometric vectors.

Dimension

Theorem

Any two bases of a vector space V have the same number of vectors i.e. if $B_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B_2 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ are bases for V then $m = n$.

Proof.

□

Let V = vector space. We say V is **finite dimensional** if it has a (finite) basis B. In this case, the number of basis vectors is called the **dimension** of V and is denoted by $\dim V$.

$$\dim V = \text{no. elements in } B$$

Dimensions of some common vector spaces

Example

$\dim(\mathbb{R}^n) = n$.

since the standard basis for $\mathbb{R}^n$ consists of n vectors.

Example

$\dim(\mathbb{P}_n) = n + 1$.

Since

Reducing spanning sets to bases

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^m$, $A = (\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_n)$, $\mathbf{x} = (x_1 \ldots x_n)^T$.
Consider eqn

$$A\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}.$$

S is linearly independent iff $\mathbf{x} = \mathbf{0}$ is the unique soln.
If $\mathbf{x}$ has soln with $x_i \neq 0$, deleting $\mathbf{v}_i$ from S does not affect span.
Let's row reduce A to a row-echelon form U. Then

- S is linearly independent iff all columns of U are leading;
- S is linearly dependent iff at least one of the columns of U is non-leading;
- the vectors in S corresponding to the leading columns of U form a basis for span(S).

Example of reducing spanning sets to a basis

Example

Let $S = \left\{ \begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix} \right\}$.

Find a basis for $\text{span}(S)$. What's $\dim \text{span}(S)$?

Example of finding a basis cont'd

More gen

Theorem

If S is a finite spanning set for a vector space V, then S contains a subset B which is a basis for V.

Extending linearly independent sets to bases

Theorem

Let W = subspace of finite dimensional vector space V. Every linearly independent subset of W can be extended to a basis for W. In particular, W is also finite dimensional.

In theory, we keep adding a vector to our linearly independent set S, which does not belong to span(S), until we get a basis.

In practice,

Example

Let $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$. Find a basis for $\mathbb{R}^3$ containing as many of the vectors in S as possible .

Example cont'd

Solution (Continued)

Solution (Continued)

Summary of algorithms for finding bases

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_s\} \subset \mathbb{R}^m$.

Theorem

Let $A = (\mathbf{v}_1 | \ldots | \mathbf{v}_s)$. If U is a row-echelon form for A, the columns of A corresponding to leading columns in U form a basis for span(S).

Theorem

Let $A = (\mathbf{v}_1 | \ldots | \mathbf{v}_s | \mathbf{e}_1 | \ldots | \mathbf{e}_m)$. If U is a row-echelon form for A, the columns of A corresponding to leading columns in U form a basis for $\mathbb{R}^m$ and this basis contains as many vectors in S as possible.
In particular, if S is independent, the basis so formed will contain S as a subset.

Things you know if you know dim V

Theorem

For any finite dim vector space V with $\dim V = d$,

1. *Any spanning set S for V has $\geqslant d$ vectors,*
2. *Any linearly independent set S in V has $\leqslant d$ vectors,*
3. *if a spanning set S for V has only d vectors, then S is linearly independent and so forms a basis for V,*
4. *if a linearly independent subset S of V has d vectors, then S also spans V and thus form a basis for V.*

For proofs see Algebra Notes 6.6 – Theorem 3.

Examples on facts using dimension

Example

Can 5 vectors in $\mathbb{R}^4$ be linearly independent?

Solution

$\dim(\mathbb{R}^4) = 4$.
Thus, any linearly independent set in $\mathbb{R}^4$ has at most 4 vectors.
No, 5 vectors in $\mathbb{R}^4$ cannot be linearly independent.

Example

Are there any spanning set of $\mathbb{P}_3$ contains only 3 polynomials?

Solution

$\dim(\mathbb{P}_3) = 4$.
Thus, any spanning set of $\mathbb{P}_3$ has at least 4 vectors.
No, a set of 3 polynomials in $\mathbb{P}_3$ cannot be a spanning set of $\mathbb{P}_3$.

Example

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal set in $\mathbb{R}^3$. Show that B is a basis.

Solution

We begin by showing that B is linearly independent.

Now $\dim \mathbb{R}^3 = 3$ *so by our thm, the linearly independent set B which has 3 vectors must also be a basis.*