MATH1141 JUNE 2013

2. i) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0. \end{cases}$$

a) Explain why f is differentiable everywhere and determine $f'(x)$.

b) Explain why the function g defined by $g(x) = f'(x)$ is continuous at $x = 0$.

c) Use the definition of the derivative to determine whether g is differentiable at $x = 0$.

ii) Consider the function $f(x) = \dfrac{1}{1+x}$ defined on $[0,1]$ and let P be the partition $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$.

a) Show that the lower Riemann sum $L_P(f)$ is given by

$$L_P(f) = \sum_{k=1}^{n} \frac{1}{n+k}.$$

b) Assuming that the limits of the upper and lower Riemann sums are equal, evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}.$$

iii) Let A, B and C be three points in the plane with corresponding position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

a) Let M be the midpoint of the line joining A and B. What is the position vector $\mathbf{m}$ of M?

b) Write a parametric vector equation for the line through C and M.

c) Suppose that

$$(\mathbf{b}-\mathbf{a}) \cdot (\mathbf{c}-\mathbf{a}) = \frac{1}{2}|\mathbf{b}-\mathbf{a}|\,|\mathbf{c}-\mathbf{a}| \quad \text{and} \quad (\mathbf{c}-\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b}) = \frac{1}{2}|\mathbf{c}-\mathbf{b}|\,|\mathbf{a}-\mathbf{b}|.$$

Explain why the triangle ABC is equilateral.

iv) Consider the system of equations

$$x + y - z = 2 \qquad (1)$$
$$x - y + 3z = 6 \qquad (2)$$
$$x^2 + y^2 + z^2 = 10 \qquad (3)$$

[Note that equation (3) is NOT linear.]

a) Give, in parametric vector form, the set of points which satisfy the first two equations (that is, (1) and (2)).

b) Describe this solution set geometrically.

c) Using the answer to (a), or otherwise, find all the points which satisfy all three equations.

3. i) Find the shortest distance from the plane

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}$$

to the point $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$.

ii) Let $p(z) = z^4 - z^3 - z^2 - z + 2$. Denote the roots of p by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, where α_1 is an **integer**.

a) Find the value of α_1.

b) Given that at least one of the roots of p is not real, deduce how many of the roots are real.

c) By considering the sum of the roots, or otherwise, prove that at least one of the roots has negative real part.

d) Prove that $|\alpha_j| > \frac{1}{2}$ for $j = 1, 2, 3, 4$.

iii) Which of the following statements are true **for all** non-zero 2×2 matrices $A, B, C \in M_{2,2}(\mathbb{R})$? For those statements which are not always true, give a counterexample.

a) $AB = BA$.

b) $\det(AB) = \det(BA)$.

c) If $\det(AB) = \det(AC)$ then $\det(B) = \det(C)$.

d) If $AB = AC$ then $B = C$.

iv) a) Define what it means for a set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ to be an **orthonormal set** in $\mathbb{R}^n$.

b) Let M be the matrix whose columns consist of the n orthonormal vectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$. By considering $M^T M$ or otherwise, find, with reasons, all possible values for $\det(M)$.

4. i) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \int_0^{x^3} e^{-t^2}\,dt.$$

a) Determine, with reasons,

$$\lim_{t\to\infty} t^2 e^{-t^2}.$$

b) Does the improper integral

$$I = \int_0^{\infty} e^{-t^2}\,dt$$

converge? Give reasons for your answer.

c) Find all critical points and asymptotes of f.

d) Carefully sketch the graph of f, clearly indicating the above information and any other relevant features.

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3. i) Let $g(x) = 3x - \cos 2x - 1$, $x \in \mathbb{R}$. Explain why g has a differentiable inverse function $h = g^{-1}$ and calculate $h'(-2)$.

   ii) a) State carefully the Mean Value Theorem.

   b) Use the Mean Value Theorem to prove that if $a < b$ then

$$0 < \tan^{-1} b - \tan^{-1} a \leq b - a.$$

   c) Using (b) or otherwise, prove that the improper integral

$$I = \int_1^{\infty} \tan^{-1}\left(t + \frac{1}{t^2}\right) - \tan^{-1} t \, dt$$

   converges.

   iii) Use the ϵ-M definition of the limit to prove that

$$\lim_{x \to \infty} \frac{e^x}{\cosh x} = 2.$$

   iv) Consider the polar curve $r = 1 + \cos 4\theta$.

   a) Determine the values of $\theta \in [0, 2\pi]$ for which r has the smallest and largest values.

   b) Hence, or otherwise, sketch this polar curve. (You are not required to find the slope.)

   v) For $x > 0$, let $f(x) = x^{x \ln x}$.

   a) Evaluate $f'(x)$.

   b) Determine the values of x for which $f'(x) > 0$ and the values of x for which $f'(x) < 0$.

   c) Given that $\lim_{x \to 0^+} f(x) = 1$, sketch the graph $y = f(x)$ for $0 \leq x \leq 2$.

4. i) Find the conditions on b_1, b_2, b_3 which ensure that the following system has a solution.

$$\begin{array}{rcrcrcl} 2x & & & - & 4z & = & b_1 \\ 3x & + & y & - & 2z & = & b_2 \\ -2x & - & y & & & = & b_3 \end{array}$$

   ii) Let I, J and K be the points in $\mathbb{R}^3$ whose position vectors are the three standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ respectively.

   By considering vectors of the form $\begin{pmatrix} x \\ x \\ x \end{pmatrix}$, find the position vector of a point A, not the origin, such that the distances from A to I, J and K are all 1.

   iii) Consider the complex matrix $A = \begin{pmatrix} 2 & i \\ 1+i & \alpha \end{pmatrix}$.

   a) Find A^{-1} in the case when $\alpha \in \mathbb{R}$.

   b) Find all values of $\alpha \in \mathbb{C}$ for which $\det(A^2) = -1$.

   iv) You may assume that $(z^9 - 1) = (z^3 - 1)(z^6 + z^3 + 1)$.

   a) Explain why the roots of $z^6 + z^3 + 1 = 0$ are $e^{\pm \frac{2\pi i}{9}}, e^{\pm \frac{4\pi i}{9}}, e^{\pm \frac{8\pi i}{9}}$.

b) Divide $z^6 + z^3 + 1$ by z^3 and let $x = z + \frac{1}{z}$.
Find a cubic equation satisfied by x.

c) Deduce that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$.

v) The norm $\|M\|$ of an $n \times n$ matrix M is the maximum value that $|M\mathbf{u}|$ takes for all unit vectors $\mathbf{u} \in \mathbb{R}^n$.

a) Show that for any vector $\mathbf{x} \in \mathbb{R}^n$,

$$|M\mathbf{x}| \leq \|M\||\mathbf{x}|.$$

b) Suppose that M and N are any two $n \times n$ matrices. By considering $MN\mathbf{u}$, or otherwise, show that

$$\|MN\| \leq \|M\|\|N\|.$$

c) What is the norm of the matrix $\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$