Chapter 4: Linear Equations

Daniel Chan

UNSW

Term 1 2022

An Ancient Chinese Problem

The following maths problem was taken from the Chinese maths text "Nine Chapters on the Mathematical Arts" ca. 200CE

Problem

One pint of good wine costs 50 gold pieces, one pint of bad only 10. If two pints are bought for 30 gold pieces, how much of each wine is bought?

The Chinese knew this is a simple algebra problem. If x, y are the amount (in pints) of good and bad wine bought then

$$x + y = 2 \tag{1}$$
$$50x + 10y = 30 \tag{2}$$

Kindergarten solution

We review the solution, as we wish to generalise this later.

Key Eliminate variables. This can be done in lots of ways but the method we'll generalise is

Subtract 50 $\times$ eqn (1) from eqn (2) to get

$$-40y = -70 \implies y = \frac{7}{4}.$$

Now obtain x by using this and eqn (1) (or (2))

$$x = 2 - y = \frac{1}{4}.$$

Linear equations

A *linear equation* in n unknowns $x_1, \dots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, \dots, a_n, b$ are given real numbers.

E.g. The name linear comes from

We are interested in solving several of these simultaneously as in the ancient Chineses problem. A system of m linear equations in n variables is

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

Point normal form for lines in $\mathbb{R}^2$

Just as there are point normal forms for planes in $\mathbb{R}^3$, there are point normal forms for lines in $\mathbb{R}^2$ (can you see the general pattern??)

Let $L \subset \mathbb{R}^2$ be the line in $\mathbb{R}^2$ passing through the point $\boldsymbol{a}$ with normal vector $\boldsymbol{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$.

L is the set of points $\boldsymbol{x}$ satisying the point normal form

$$\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{a}) = 0 \qquad \iff \qquad n_1 x_1 + n_2 x_2 = \boldsymbol{n} \cdot \boldsymbol{x} = \boldsymbol{n} \cdot \boldsymbol{a}.$$

Intersecting 2 lines in $\mathbb{R}^2$

Consider solving a system of 2 eqns in 2 unknowns x, y:

$$a_{11}x + a_{12}y = b_1 \tag{3}$$
$$a_{21}x + a_{22}y = b_2 \tag{4}$$

Plotting solutions in the xy-plane allows us to understand the nature of solutions better. Assume both eqns are non-zero (otherwise ??) so eqns (3), (4) define lines L_1, L_2 and $L_1 \cap L_2$ represents the simultaneous soln.
On geometric grounds, we see there are ?? possibilities

Let's verify this algebraically!

2 linear equations in 2 unknowns: algebraic version

- $L_1 = L_2$ iff eqn (3) and (4) are scalar multiples of each other. Simultaneous soln same as the soln to one of them.
- $L_1 \parallel L_2$ but not equal. this means that the normals are parallel so $\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = r \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$, for some $r \in \mathbb{R}$. Then $(3) - r \times (4)$ gives

$$0 = b_1 - rb_2 \neq 0 \qquad \text{WHY?}$$

  so there are no solns.
- $\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix}, \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$ are not parallel. Our procedure gives a unique soln. This is easy to see from any e.g. such as in the ancient Chinese problem.

Intersecting 3 planes in $\mathbb{R}^3$

Here are some possibilities for intersecting 3 planes $P_1, P_2, P_3 \subset \mathbb{R}^3$.

- The planes intersect in a point.
- There is no intersection as the three are parallel and distinct.
- P_1, P_2 intersect in a line which is parallel to P_3. There's no intersection.
- $P_1 = P_2$ intersect P_3 in a line so the three intersect in a line.
- etc

There are heaps more cases, which makes it hard to analyse. We need make systematic, our method of solving simultaneous solns and understanding the set of solns.

This procedure, described in the "Nine Chapters" and re-discovered by Gauss is called *Gausssian elimination*.

Aside Gauss developed it to solve a system of 6 linear eqns in 6 unknowns and hence compute the orbit of the dwarf planet Ceres. Look up Stigler's law on Wikipedia.

Augemented matrix form

We first abbreviate our system of m linear equations in n variables:

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

to *augmented matrix form*:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right).$$

Here the notation omits the unknowns $x_1, \ldots, x_n$ which must be understood from the context (or irrelevant).

Example: augmented matrix form

E.g. Write the augmented matrix form for the system

$$\begin{cases} 6x_1 & - & 2x_2 & + & 5x_3 & = & 7 \\ 3x_1 & + & x_2 & + & 2x_3 & = & 0 \end{cases}$$

Solution.

E.g Write the system of equations with augmented matrix form
$\left(\begin{array}{ccc|c} 2 & 1 & 1 & -3 \\ 3 & 5 & 0 & 7 \end{array}\right)$.

Matrices

A *matrix* is just a rectangular array of numbers, like

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 5 \end{pmatrix}.$$

We would say that this is a 2×3 matrix as it has 2 rows and 3 columns.

An $m \times 1$ matrix just looks like a vector $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix} \in \mathbb{R}^m$

The augmented matrix consists of the *coefficient matrix* (on the left of the line) augmented by the right-hand-side vector. We can write $\left(\begin{array}{ccc|c} 2 & 1 & 1 & -3 \\ 3 & 5 & 0 & 7 \end{array}\right)$ as $(A|\boldsymbol{b})$ where $A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 5 & 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$.

Simultaneous equations as a single vector equation

Any system of (not necessarily linear) simultaneous eqns can be written as a single **vector** eqn.
The above eqn $(A|\boldsymbol{b})$ can be re-written as

$$\begin{pmatrix} -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 + x_3 \\ 3x_1 + 5x_2 \end{pmatrix} = x_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or perhaps as

$$x_1 \boldsymbol{a}_1 + x_2 \boldsymbol{a}_2 + x_3 \boldsymbol{a}_3 = \boldsymbol{b}.$$

where $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are the *columns* of A.

That is, solving this system is equivalent to asking:

How can we write $\boldsymbol{b}$ as a linear combination of the vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$?

Examples: different forms for linear equations

Q The following vector equation is equivalent to which system of linear equations and which augmented matrix?

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

Q Write a vector equation equivalent to the following augmented matrix.

$$\left(\begin{array}{cc|c} 1 & 0 & -2 \\ 2 & e^2 & \pi \end{array} \right)$$

Equivalent systems of linear equations

We say a system of linear eqns is *consistent* if it has at least one solution and *inconsistent* otherwise.

E.g. $x + y = 2, 2x + 2y = 5$ is

Definition

Two systems of linear equations $(A|\boldsymbol{b}), (A'|\boldsymbol{b}')$ are said to be *equivalent* if they have exactly the same set of solutions.

The following define equivalent systems of linear equations

$$\begin{aligned} x + y &= 2 & \qquad x + y &= 2 \qquad (5) \\ 2x + 3y &= 5 & \qquad 3x + 4y &= 7 \qquad (6) \end{aligned}$$

Why? Note that the 2nd system is gotten from the 1st by adding eqn (5)LHS to eqn (6)LHS. Similarly, the 1st system
The systems are equivalent because

Elementary row operations

To solve $(A|\boldsymbol{b})$, we pass to an equivalent system of eqns where one of the eqns has only 1 unknown. We can solve this and then by induction on the number of variables, should be done.

Q How do you pass to an equivalent system of linear equations?
Any of the following procedures will yield an equivalent system:

Elementary row operations

- Swap the order in which you write the equations!
- Multiply any equation by a **nonzero** constant. (Why non-zero?)
- Add a multiple of one equation to another equation. For example, you can replace

  equation (5)

  with

  equation $(5) + 7 \times$ equation (6).

These are called the *elementary row operations*.

This is all easier to do if we use the augmented matrix form! We just replace the word eqn above with row of the augmented matrix.

Notation for EROs

We record the elementary row operation on an arrow between the old system and the new. In each case below, the system on the right is equivalent to the left.

$$\left(\begin{array}{rrr|r} 0 & 2 & 3 & 5 \\ -1 & 3 & 1 & 6 \\ 2 & 4 & 7 & 8 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{rrr|r} -1 & 3 & 1 & 6 \\ 0 & 2 & 3 & 5 \\ 2 & 4 & 7 & 8 \end{array}\right).$$

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ -1 & -4 & 1 & 6 \\ 2 & 10 & 7 & 8 \end{array}\right) \xrightarrow{R_2 = R_2 + R_1} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ 0 & -2 & 4 & 11 \\ 2 & 10 & 7 & 8 \end{array}\right)$$

$$\left(\begin{array}{rrrr|r} 1 & 2 & 3 & 4 & 8 \\ 0 & 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 4 & 8 \end{array}\right) \xrightarrow[R_3 = \frac{1}{4}R_3]{R_2 = \frac{1}{3}R_2} \left(\begin{array}{rrrr|r} 1 & 2 & 3 & 4 & 8 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right).$$

Row echelon form: motivation

We wish to apply a series of EROs until we arrive at a system of linear eqns we can solve!

Q Which systems are these?

A Ones like the bottom right one on the previous slide. Remember, that augmented matrix is just saying

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 8 \\ & & & & x_3 & + & 2x_4 & = & 3 \\ & & & & & & x_4 & = & 2 \end{array}$$

Starting from the bottom, you can first solve x_4, and then moving up solve x_3 and then (see later) x_2, x_1.

Such a system is said to be in *row-echelon form*. Solving it by starting from the bottom eqn and progressively moving up is called *back substitution*.

Row echelon form

In any matrix

1. a *leading row* is one which is not all zeros,
2. the *leading entry* in a leading row is the first (i.e. leftmost) non-zero entry,
3. a *leading column* is a column which contains the leading entry for some row.

Row echelon form

A matrix is said to be in *row-echelon form* if

1. all leading rows are above all non-leading rows (so any all-zero rows are at the bottom of the matrix), and
2. in every leading row, the leading entry is further to the right than the leading entry in any row higher up in the matrix.

If you remove the zeros, row echelon form gives an upside-down staircase shape.

A matrix is in *reduced row echelon form* if it 1) is in row echelon form, 2) all leading entries are 1 and, 3) all entries above leading entries are 0.

Examples: leading columns, row echelon form

E.g. Check the following is in row echelon form and solve

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 2 & 4 \end{array}\right)$$

E.g.

$$\begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 2 & \pi \\ 0 & 0 & e & 0 \end{pmatrix}$$

Back-substitution

We can solve systems of linear eqns if they are in row echelon form as in the following e.g.

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 8 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \qquad \begin{array}{c} (R1) \\ (R2) \\ (R3) \end{array}$$

From $(R3)$, $x_4 = 2$. Back substituting this into $(R2)$ you get $x_3 =$.

The convention is that you set the variable for any non-leading column to be an arbitrary parameter.

So, let $x_2 = \lambda$. Then

$$x_1 = 8 - 2\lambda - 3 \times (-1) - 4 \times 2 = 3 - 2\lambda.$$

Vector form of the solution

It is instructive to write the soln in vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 - 2\lambda \\ \lambda \\ -1 \\ 2 \end{pmatrix} =$$

This shows the solution set is a line in $\mathbb{R}^4$ in the direction

Upshot This argument shows that you will have as many parameters $\lambda_1, \dots, \lambda_j$ as you have non-leading columns. Geometrically, the solution set is j-dimensional. We usually expect you to write your answer in vector form as above.

Gaussian elimination: easy example

There are many sequences of EROs you can apply to obtain row echelon form (and hence solve systems of linear equations). One algorithm is easily seen in the following example. It corresponds to eliminating variables systematically.
First use $R_j = R_j + \alpha_j R_1$, to get zeros in the first column of every row (except row 1!):

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ -1 & -4 & 1 & 6 \\ 2 & 10 & 7 & 8 \end{array}\right) \xrightarrow{R_2 = R_2 + ?} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ 0 & -2 & 4 & 11 \\ & & & \end{array}\right)$$

Now rows 1 and 2 are looking fine. Essentially we eliminated x_1 from eqns in R2, R3. Next use ERO $R_j = R_j + \beta_j R_2$ to get zeros in the 2nd column of every row (except rows 1 and 2):

$$\xrightarrow{R_3 = R_3 - ?} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 5 \\ 0 & -2 & 4 & 11 \\ 0 & 0 & & \end{array}\right)$$

If the matrix were bigger, you would just keep going zeroing one column in turn, until the matrix is in row-echelon form.

Pivots

The above algorithm is fine until you meet a matrix like

$$\left(\begin{array}{rrr|r} 0 & 2 & 3 & 5 \\ 2 & -2 & 4 & 1 \\ 1 & 3 & 2 & 6 \end{array}\right) \text{ or } \left(\begin{array}{rrr|r} 0 & 0 & 3 & 5 \\ 0 & 0 & 4 & 1 \\ 0 & 1 & 2 & 6 \end{array}\right)$$

Here you need to swap the first row with the second or third.
In general

Pivot elements

1. Go to the 1st (from left) nonzero column called the *pivot column*.
2. Go down that column to the 1st nonzero entry called the *pivot element*.
3. Swap that row, called the *pivot row* with the first (unless of course it is the first).

Gaussian elimination: complete algorithm

Algorithm

To solve $(A|\boldsymbol{b})$:

1. Select the pivot element.
2. Swap the pivot row to the top if necessary.
3. Reduce to zero all the entries below the pivot element using EROs.
4. Repeat steps 1, 2 and 3 on the submatrix of rows and columns to the right and below the pivot element...recursively until you run out of pivot elements!
5. When you arrive at row-echelon form, use back-substitution to solve.

Example. Solve

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -2 \\ -1 & 1 & -1 & 1 & 2 \\ 1 & 0 & 2 & 1 & 1 \\ 3 & -1 & 5 & 1 & 0 \end{array}\right)$$

Gaussian elimination example

Application: geometry

Q Do the lines

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R} \qquad \boldsymbol{x} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \quad \mu \in \mathbb{R}$$

intersect?

You can turn this into a vector equation in λ and μ:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$\lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix}.$$

Application cont'd

As we saw before, this corresponds to a system of linear eqns which, in augmented matrix form is

$$\left(\begin{array}{cc|c} -1 & -2 & -1 \\ 1 & 1 & -3 \\ -1 & -2 & -2 \end{array}\right) \xrightarrow[R_3 = R_3 - R1]{R_2 = R_2 + R1} \left(\begin{array}{cc|c} -1 & -2 & -1 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{array}\right)$$

For this, and many other problems, what you want to know is

'Does a solution exist?', not 'What is the solution?'

You can read this straight off the row-echelon form!

The system is inconsistent because the bottom row gives the inconsistent eqn

$$0\lambda + 0\mu = -1.$$

No intersection so the lines are skew.

Remark: An inconsistent eqn corresponds to a row in the augmented matrix of the form

$$(0\ 0 \ldots 0|b) \text{ where } b \neq 0.$$

For $(U|\boldsymbol{y})$ in row echelon form, this occurs precisely when $\boldsymbol{y}$ is a leading column as above.

Nature of solutions and row echelon form

Back-substitution readily gives

Theorem

Suppose that the system $(A|\boldsymbol{b})$ has equivalent row-echelon form $(U|\boldsymbol{y})$.

- The system has no solution iff $\boldsymbol{y}$ is a leading column iff it has an inconsistent row.
- If $\boldsymbol{y}$ is not a leading column then
  1. The system has a unique solution if U has no non-leading columns, and
  2. The system has infinitely many solutions if U has non-leading columns. There is one arbitrary parameter for each nonleading column.

E.g. Describe geometrically, the solution sets to

$$\left(\begin{array}{rrr|r} -1 & 3 & 1 & 6 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 7 & 8 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{rrr|r} -1 & 3 & 1 & 6 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Underdetermined equations

Question. Can the system

$$\left(\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & -2 \\ 2 & -2 & -1 & 3 & 7 \\ -2 & -3 & 3 & 8 & 5 \end{array}\right)$$

have a unique solution?

Solution. In general, if you have more variables than equations you cannot have a unique soln. Indeed, in the row-echelon form $(U|\boldsymbol{y})$,

$$\text{no. leading columns} = \text{no. leading rows} < \text{no. columns in } U$$

so there must be a non-leading column in U.

If $\boldsymbol{y}$ is leading then

If $\boldsymbol{y}$ is non-leading then

Describing spans in cartesian form

Q Which $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3$ are in the span of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$?

A As a vector eqn, this is asking whether there exist $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} +$$

In augmented matrix form this is

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & b_1 \\ 2 & -1 & 3 & b_2 \\ 3 & -2 & 4 & b_3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & b_1 \\ 0 & -3 & -3 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + \frac{b_1}{3} - \frac{5b_2}{3} \end{array}\right)$$

Describing Span cont'd

We need to know whether this system has a solution.
The right-hand side vector here is a leading column — unless

$$b_3 + \frac{b_1}{3} - \frac{5b_2}{3} = 0. \tag{$*$}$$

$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is a linear combination of the three vectors precisely when $(*)$ holds.

The span of the 3 vectors is the plane in $b_1 b_2 b_3$-space defined by the cartesian eqn $(*)$.

Can you see what is happening geometrically?

Solving simultaneous equations: functional viewpoint

We often solve eqns like $\sin x = \frac{1}{2}$ or more generally $f(x) = c$ for some $f : \mathbb{R} \longrightarrow \mathbb{R}$ and constant $c \in \mathbb{R}$. Solving simultaneous eqns can be put in this framework if we allow vector-valued functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Consider $a_1, \ldots, a_n \in \mathbb{R}$ and the function $l : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$l(\boldsymbol{x}) = a_1 x_1 + \ldots + a_n x_n.$$

This is a *scalar valued linear function* (with vector inputs). $l(\boldsymbol{x}) = b$ is a linear eqn. Given a vector $\boldsymbol{x} \in \mathbb{R}^n$ and an $m \times n$-matrix

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \text{ we define } A\boldsymbol{x} = \begin{pmatrix} a_{11}x_1 + \ldots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n \end{pmatrix}.$$

$l : \mathbb{R}^n \longrightarrow \mathbb{R}^m : \boldsymbol{x} \mapsto A\boldsymbol{x}$ i.e. defined by $l(\boldsymbol{x}) = A\boldsymbol{x}$ is an example of a *vector-valued linear function*.
The system of linear eqns corresponding to $(A|\boldsymbol{b})$ is equivalent to the vector equation $A\boldsymbol{x} = \boldsymbol{b}$. We call $A\boldsymbol{x}$ the *matrix-vector* product of A and $\boldsymbol{x}$.

Matrix-vector product: examples

Q Calculate

$$\begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Q Let $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Describe the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : \boldsymbol{x} \mapsto A\boldsymbol{x}$ as a mapping from the plane to itself. Hence solve $f(\boldsymbol{x}) = \boldsymbol{x}$. We'll see much more of this next semester.

Distributive law for matrix-vector product

Part of the reason we use the product notation/terminology is the

Distributive Law

Suppose that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and A is an $m \times n$-matrix. Then

$$A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y}.$$

Why? For ease of notation, suppose $n = 2$ so $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

$$\begin{aligned} A(\boldsymbol{x} + \boldsymbol{y}) = A\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} &= \begin{pmatrix} a_{11}(x_1 + y_1) + a_{12}(x_2 + y_2) \\ \vdots \\ a_{m1}(x_1 + y_1) + a_{m2}(x_2 + y_2) \end{pmatrix} \\ &= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 \end{pmatrix} + \begin{pmatrix} a_{11}y_1 + a_{12}y_2 \\ \vdots \\ a_{m1}y_1 + a_{m2}y_2 \end{pmatrix} = A\boldsymbol{x} + A\boldsymbol{y} \end{aligned}$$

Ex Prove that if $\lambda \in \mathbb{R}$, $A(\lambda \boldsymbol{x}) = \lambda(A\boldsymbol{x})$.

Linearity

Next semester, we will study the concept of linearity in great generality. For now, we say a function $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is *linear* if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \lambda \in \mathbb{R}$ we have

$$T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y}), \qquad T(\lambda \boldsymbol{x}) = \lambda T(\boldsymbol{x}).$$

Last slide $\implies$ the function $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m : \boldsymbol{x} \mapsto A\boldsymbol{x}$ is linear. Next semester, we'll see all linear maps $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ have this form.

E.g. $T\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y$

Warning Our defn conflicts with the one in calculus! $f(x) = ax + b$ is linear iff

Sometimes, we call such functions *affine linear*.

Homogeneous equations: abstract view

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear function and $\boldsymbol{b} \in \mathbb{R}^m$. We consider the *linear equation* $T\boldsymbol{x} = \boldsymbol{b}$ and solve for $\boldsymbol{x}$.
We say $T\boldsymbol{x} = \boldsymbol{b}$ is *homogeneous* if $\boldsymbol{b} = \boldsymbol{0}$, and is *inhomogeneous* otherwise.

Proposition

Suppose $\boldsymbol{x}_1, \dots, \boldsymbol{x}_r$ are solutions to the homogeneous eqn $T\boldsymbol{x} = \boldsymbol{0}$.

1. $\boldsymbol{x} = \boldsymbol{0}$ is a homogeneous soln, (existence).
2. $\boldsymbol{x}_1 + \boldsymbol{x}_2, \lambda\boldsymbol{x}_1$ are homog solns for any $\lambda \in \mathbb{R}$.
3. Any linear combination $\lambda_1\boldsymbol{x}_1 + \ldots + \lambda_r\boldsymbol{x}_r$ is a homog soln.

Proof. For (1) just calculate.
2) $\implies$ 3) by induction.
For 2) just calculate:

$$T(\boldsymbol{x}_1 + \boldsymbol{x}_2) = T\boldsymbol{x}_1 + T\boldsymbol{x}_2 = \boldsymbol{0} + \boldsymbol{0} = \boldsymbol{0}.$$

Similarly, $T(\lambda\boldsymbol{x}_1) =$
Terminology We express (2) by saying that the set of homogeneous solns is *closed under addition and scalar multiplication*. We express (3) by saying homog solns are *closed under linear combinations*.

Inhomogeneous equations: abstract view

The solutions to an inhomogeneous linear eqn $T\boldsymbol{x} = \boldsymbol{b}$ are related to the solns of the corresponding homogeneous eqn $T\boldsymbol{x} = \mathbf{0}$, as long as solns exist!

Proposition

Let $\boldsymbol{x} = \boldsymbol{x}_p$ be a soln to $T\boldsymbol{x} = \boldsymbol{b}$.

1. Then $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_h$ is also a solution for any homog soln $\boldsymbol{x}_h$ to $T\boldsymbol{x} = \mathbf{0}$.
2. Every soln to $T\boldsymbol{x} = \boldsymbol{b}$ has this form.

Remark The importance of the proposition is that if you know a particular inhomogeneous soln, and the **general** homogeneous soln, then you know the general inhomogeneous soln too.

Proof. 1)

2) Let $\boldsymbol{x}_0$ be a soln to $T\boldsymbol{x} = \boldsymbol{b}$. Let $\boldsymbol{x}_h = \boldsymbol{x}_0 - \boldsymbol{x}_p$ so $\boldsymbol{x}_0 = \boldsymbol{x}_p + \boldsymbol{x}_h$. It suffices to show $\boldsymbol{x}_h$ is a homog soln:

Corollary If $\boldsymbol{x} = \mathbf{0}$ is the unique homogeneous soln, then an inhomogeneous soln is unique (assuming it exists).

Relating the abstract view with result from Gaussian elimination

E.g. Consider the linear eqn $A\boldsymbol{x} = \boldsymbol{b}$ corresponding to the reduced row echelon augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 8 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Assigning parameters $x_2 = \lambda, x_4 = \mu$, back-substution gives the general soln as

$$\boldsymbol{x} = \qquad\qquad = \begin{pmatrix} 8 \\ 0 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

Setting $\lambda = \mu = 0$ gives the particular soln $\begin{pmatrix} 8 \\ 0 \\ 3 \\ 0 \end{pmatrix}$ and the propn last slide $\Longrightarrow$

the general homog soln is the span of

Application: polynomials

Question

Fix real numbers a, b, c. Are there any quadratics $p(x) = a_0 + a_1 x + a_2 x^2$ which satisfy the following conditions

$$p(1) = a, \quad p'(1) = b, \quad p''(1) = c?$$

A The conditions give linear conditions on the coefficients. Now

$$\begin{aligned} a &= p(1) = a_0 + a_1 + a_2 \\ b &= p'(1) = a_1 + 2a_2 \\ c &= p''(1) = 2a_2 \end{aligned}$$

Application: Geometry

Q Is the line with parametric equation $\boldsymbol{x} = \begin{pmatrix} -9 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ parallel to the plane through the points with coords $(0,0,0), (1,0,1), (2,-1,2)$?

A The question is equivalent to asking if $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ lies in the span of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

i.e. can we solve

Application: Differential equations

Q Find all polynomial functions $f(x) = \sum_{j=o}^{n} a_j x^j$ which satisfy the ODE

$$(x+2)f'(x) - f(x) = 1$$

A LHS $= (x+2)\sum_{j=1}^{n} j a_j x^{j-1} - \sum_{j=o}^{n} a_j x^j = \sum j a_j x^j + 2\sum j a_j x^{j-1} - \sum a_j x^j$

so equating co-eff of x^j:

Hence $0 = a_2 = \ldots = a_n, a_1 = \lambda, a_0 =$

$\therefore f(x) =$

High faluting viewpoint of ODEs

The "differential operator" $f(x) \mapsto (Tf)(x) := (x+2)f'(x) - f(x)$ is linear!
Why?

A particular soln to "linear" eqn $(Tf)(x) = 1$ is $f(x) =$
The "homogeneous" soln $y = f(x)$ is found by solving $(x+2)\frac{dy}{dx} - y = 0$.

Modern economic problem

The Republic produces 3 types of X-wing fighters: 1) 1 person, 2) 2 person and 3) 4 person. Production involves 3 plants operated by Gungans, Ewoks and Wookiees, which deal with a) weapons manufacture, b) engine manufacture and c) assembly respectively. The time (in hours) required to produce each type of fighter is given below

	1 person	2 person	4 person
Gungans	1	2	3
Ewoks	2	3	4
Wookiees	2	4	8

The Gungans will only work 20 hours a week, the Ewoks 33 and the Wookiees 44. If the production plants are to operate at full capacity, how many of each type of fighter will be produced?

A Let x_1, x_2, x_3 be the number of 1 person, 2 person and 4 person fighters produced a week.

We have one equation for each plant.

Economic problem cont'd