Chapter 3: Complex Numbers

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Term 1 2020
Philosophical discussion about numbers

In what sense is $-1$ a number? DISCUSS

Is $\sqrt{-1}$ a number?

Not a REAL number.

Why not then a non-real number? After all, $\sqrt{-1}$ exists as an expression, and as such it pops up all the time when you solve enough equations EVEN IF you are only interested in REAL numbers (see later).

OK. Let's extend our number system by pretending $\sqrt{-1}$ is a number which we'll denote as usual by $i$, and see what happens.

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Well if $i$ is a number, then surely so is $3i$ and $2 + 3i$. In fact, for any $a, b, c, d \in \mathbb{R}$, $a + bi, c + di$ are numbers too, surely.

But then $(a + bi) + (c + di)$ is a number! That's OK, it must be one we've seen before $(a + c) + (b + d)i$.

But also $(a + bi)(c + di)$ is a number(??). I guess it ought to be $(a + bi)(c + di) = ac + bci + ad + bdi = (ac - bd) + (bc + ad)i$ since $i^2 = -1$.

We've seen this number before. Q

When does $a + bi = c + di$?

A Then $(a - c)^2 = (d - b)^2i^2 = -(d - b)^2$ which occurs precisely when $a = c$ and $b = d$. (WHY?)

Major Question If we keep playing this game blindly, using our usual rules of arithmetic, will we ever end up proving absurd statements like $1 = 0$?
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If we keep playing this game blindly, using our usual rules of arithmetic, will we ever end up proving absurd statements like \( 1 = 0 \)?
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- a multiplication rule, which assigns to any $x, y \in \mathbb{F}$ an element $xy \in \mathbb{F}$. 
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**Definition**

A *field* is the data consisting of a non-empty set $F$ together with

- an *addition rule* $+$, which assigns to any $x, y \in F$ an element $x + y \in F$.
- a *multiplication rule*, which assigns to any $x, y \in F$ an element $xy \in F$.

such that the axioms on the following page hold.
Field axioms

1. **Associative Law of Addition**
   \[(x + y) + z = x + (y + z)\] for all \(x, y, z \in F\).

2. **Commutative Law of Addition**
   \[x + y = y + x\] for all \(x, y \in F\).

3. **Existence of a Zero**
   There exists an element of \(F\) (usually written as 0 & called the zero) such that \(0 + x = x + 0 = x\) for all \(x \in F\).

4. **Existence of a Negative**
   For each \(x \in F\), there exists an element \(w \in F\) (usually written as \(-x\) & called the negative of \(x\)) such that \(x + w = w + x = 0\).

5. **Associative Law of Multiplication**
   \[x(yz) = (xy)z\] for all \(x, y, z \in F\).

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   \[xy = yx\] for all \(x, y \in F\).

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   There exists a non-zero element of \(F\) (usually written as 1 & called the multiplicative identity) such that \(x1 = 1x = x\) for all \(x \in F\).

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Examples

E.g. \( F = \mathbb{R}, \mathbb{Q} \) are fields when endowed with the usual addition and multiplication of numbers for the addition and multiplication rule.

E.g. the field with 2 elements

Let \( F = \{ \text{even}, \text{odd} \} \).

Define the addition rule by

\[
\text{even} + \text{even} = \text{even}, \quad \text{even} + \text{odd} = \text{odd},
\]

and the multiplication rule by

\[
\text{even} \times \text{even} = \text{even}, \quad \text{even} \times \text{odd} = \text{even},
\]

You can check all field axioms are satisfied.

Remark

This field is very important in coding theory.
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$$\text{even} + \text{even} = \text{even}, \quad \text{even} + \text{odd} = \text{odd}, \ldots.$$ 

and the multiplication rule by

$$\text{even} \times \text{even} = \text{even}, \quad \text{even} \times \text{odd} = \text{even}, \ldots.$$ 

You can check all field axioms are satisfied.

**Remark** This field is very important in coding theory.
What’s subtraction and division?

The point of the axioms, is that this is the minimal set of assumptions to ensure you can do all the usual arithmetic in the usual way.

**Fact**
In a field \( F \), the zero, negative, one and multiplicative inverse are unique. (What’s this mean?)

The proof (omitted) is not hard, but many of you might find it strange.

Hence for \( x, y \in F \) we can define:

\[
x - y = x + (-y)
\]

and if \( y \neq 0 \),

\[
x / y = xy - 1
\]
What’s subtraction and division?

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In particular, you can subtract and divide (by non-zero field elements). To do this you need
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E.g.

Simplify the following expression in a field $x(y + z) - yx$.
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**E.g.** Simplify the following expression in a field

$$x(y + z) - yx$$
Our thought experiment suggests the following

**Definition**

A complex number is a formal expression of the form \(a + bi\) for some \(a, b \in \mathbb{R}\). In particular, two such numbers \(a + bi, a' + b'i\) are equal iff \(a = a'\), \(b = b'\) as real numbers.

**Remarks**

1. Formal means in particular, that the + is just a symbol, it doesn't mean addition (yet).
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Arithmetic of complex numbers

Definition

Given complex numbers $a + bi$, $a' + b'i$ as above, we define addition and multiplication by

\[
(a + bi) + (a' + b'i) = (a + a') + (b + b')i
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\[
(a + bi)(a' + b'i) = (aa' - bb') + (ab' + a'b)i
\]

Warning

There are two clashes of notation. What's $a + bi$ mean?

We're OK.

Theorem

The set $\mathbb{C}$ of complex numbers with the above addition and multiplication rule is a field.

Proof.

Is long and tedious but elementary. Note zero is $0 + 0i$.

This means we can perform complex number arithmetic as usual.

N.B. $\mathbb{C}$ extends the real number system since complex numbers of form $a + 0i$ add and multiply just like real numbers.
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Examples of complex arithmetic

Eg What's the negative of $a + bi$?

Eg $(5 - 7i) - (6 + i)$?

Eg Simplify $(2 + i)(1 - 3i) - 1 + 3i$
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Division

To get the inverse we need the Cool Formula: Let $z = a + bi \in \mathbb{C}$ (with $a, b \in \mathbb{R}$ of course). We define the conjugate of $z$ to be $\overline{z} = a - bi$. 

$z \overline{z} = a^2 + b^2 \in \mathbb{R} \geq 0$.

This gives the multiplicative inverse of $z$ as $z^{-1} = \frac{\overline{z}}{a^2 + b^2}$.

This is all we need since we know inverses of real numbers. Usually though, we divide as follows: E.g.
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**E.g.**
Cartesian form

A complex number $z$ written in the form $a + bi$ with $a, b \in \mathbb{R}$ is called the cartesian form (Later we'll meet the polar form).

Express $1 + i$ as a cartesian form.

Daniel Chan (UNSW)  
Chapter 3: Complex Numbers  
Term 1 2020
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Q Express $\frac{1+i}{1-i} - \frac{1-i}{1+i}$ in cartesian form.
Properties of conjugation

1. \( z \) is real iff \( z = \overline{z} \).

2. \( z = \overline{z} \).

3. \( z + w = \overline{z} + \overline{w} \) and \( z - w = \overline{z} - \overline{w} \).

4. \( zw = \overline{z} \overline{w} \) and \( (z \overline{w}) = z \overline{w} \).

5. \( \text{Re}(z) = \frac{1}{2}(z + \overline{z}) \) and \( \text{Im}(z) = \frac{1}{2}i(z - \overline{z}) \).

Proof. Easy. Write both sides out e.g.

E.g. Show that for any \( z \in \mathbb{C} \), \( (i + 5)z - (i - 5)z \) is real.
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Proposition

1. $z$ is real iff ($\iff$ if and only if) $\bar{z} = z$.
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Properties of conjugation

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**Proof.** Easy. Write both sides out e.g.

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Chapter 3: Complex Numbers

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The Argand diagram

Just as real numbers can be represented by points on the real number line, complex numbers can be represented on the complex plane (or Argand diagram) as follows.

\[ z = a + bi \]

is represented by the point with coords \((a, b) = (\text{Re} z, \text{Im} z)\).

The axes though are called the real and imaginary axes.

Adding complex numbers is by adding real and imaginary parts, i.e. coordinatewise so is represented geometrically by the addition of vectors. Similarly for subtraction.
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Polar form

Writing a complex number as \( z = x + yi \), \( x, y \in \mathbb{R} \) is called the cartesian form of \( z \). It corresponds to rectilinear coordinates.

Suppose the polar coordinates for \( z \) are given by \((r, \theta)\) as above. \( z = r \cos \theta + (r \sin \theta) i \).

**Definition**

1. The **modulus of** \( z \) is defined to be \( |z| = r = \sqrt{x^2 + y^2} \) so \( z \overline{z} = |z|^2 \).

2. If \( z \neq 0 \), an **argument for** \( z \) is any \( \theta = \text{arg} \ z \) as above i.e. so that \( \tan \theta = \frac{y}{x} \) and \( \cos \theta, \text{Re} \ z \) have the same sign. \( \theta = : \text{Arg} \ z \) is the **principal argument** if \( -\pi < \theta \leq \pi \).
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Examples: modulus and argument

E.g. Find the modulus and principal argument of $-\frac{1}{2} - \sqrt{3}i$.

E.g. Find the complex number with modulus 3 and argument $\pi/4$. 
Examples: modulus and argument

E.g. Find the modulus and principal argument of $-1 - \sqrt{3}i$. 
E.g. Find the modulus and principal argument of $-1 - \sqrt{3}i$.

E.g. Find the complex number with modulus 3 and argument $\pi/4$. 
Euler’s formula

Definition (Euler’s formula)

For $\theta \in \mathbb{R}$, we define $e^{i\theta} = \cos \theta + i\sin \theta$.

This is reasonable by

1. $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$.

2. (De Moivre’s thm) For $n \in \mathbb{Z}$, $(e^{i\theta})^n = e^{in\theta}$.

3. $\frac{d}{d\theta}(e^{i\theta}) = ie^{i\theta}$.

Proof. 2) & 3) easy omitted. We only check 1).

$(\cos \theta_1 + i\sin \theta_1)(\cos \theta_2 + i\sin \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i(\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$.

Challenge Q

What’s $i^i$?
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$(\cos \theta_1 + i \sin \theta_1) \times (\cos \theta_2 + i \sin \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2) = \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)$. 

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**Challenge Q**

What’s $i\times i$?
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$$(\cos \theta_1 + i \sin \theta_1) \times (\cos \theta_2 + i \sin \theta_2)$$

$$= \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i(\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)$$

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### Definition (Euler’s formula)

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### Proof.

2) & 3) easy omitted. We only check 1).

\[
\begin{align*}
(c_1 + j c_2) \times (c_3 + j c_4) &= c_1 c_3 - j c_1 c_4 + j c_2 c_3 - c_2 c_4 \\
&= \cos(c_1 + c_3) + j (\sin(c_1 + c_3) + \sin(c_2 + c_4)) \\
&= \cos(c_1 + c_3) + j \sin(c_1 + c_3).
\end{align*}
\]

### Challenge Q

What’s \( i^i \)?
The polar form of $z$ is $z = re^{i\theta}$ where $r = |z|$ and $\theta$ is an argument of $z$.

Our formulas above give $r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$, $r_1 e^{i\theta_1} - 1 = r_1 e^{-i\theta_1}$.

Geometrically, this says that when you multiply complex numbers, you multiply the moduli and add the arguments.

Inverting inverts the modulus and negates the argument.

$|z_1 z_2| = |z_1||z_2|$, $|z_1 - 1| = |z_1| - 1$, $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2k\pi$, $\text{Arg}(z_1 - 1) = -\text{Arg}(z_1)$ unless where $k \in \mathbb{Z}$ is chosen so that

E.g. Find the exact value of $\text{Arg}(1+i \sqrt{3} i)$. 

Daniel Chan (UNSW)
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The polar form of $z$ is $z = re^{i\theta}$ where $r = |z|$ and $\theta$ is an argument of $z$. Our formulas above give

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Arithmetic of polar forms

The **polar form** of $z$ is $z = re^{i\theta}$ where $r = |z|$ and $\theta$ is an argument of $z$. Our formulas above give

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**E.g.** Find the exact value of $\text{Arg}\frac{1+i}{1+\sqrt{3}i}$. 
Let $z \in \mathbb{C}$ have $|z| = 1$. Show that $w = i - z + z$ is purely imaginary in the sense that $\text{Re} \, w = 0$. Interpret the result geometrically.
Q Let $z \in \mathbb{C}$ have $|z| = 1$. Show that $w = \frac{i - z}{i + z}$ is purely imaginary in the sense that $\text{Re}w = 0$. Interpret the result geometrically.
Square roots of complex numbers

E.g. Find the complex square roots \( \pm z \) of 16 \(-30i\).
E.g. Find the complex square roots $\pm z$ of $16 - 30i$
Quadratic formula

E.g.

Solve $z^2 + (1 + i)z + (-4 + 8i) = 0$. 
E.g. Solve $z^2 + (1 + i)z + (-4 + 8i) = 0$. 
In the 16th century, Ferro, Tartaglia, Cardano, and others discovered how to solve cubics. The general cubic equation

$$z^3 + pz = q$$

has solutions

$$z = \sqrt[3]{q^2 + \sqrt{q^2 + \frac{p^3}{27}}} \pm \sqrt[3]{q^2 - \sqrt{q^2 + \frac{p^3}{27}}}.$$ 

Let's use this to solve $z^3 - z = 0$ (which we know has solutions $0, 1, -1$).

A bizarre fact: If there are 3 real roots, then the formula above always involves non-real numbers.

Moral to this story: Even if you only ever cared about real numbers, complex numbers naturally arise.

An even more bizarre fact: There's a similar formula for quartics, but no such exists for higher degree (see Galois theory course MATH5725).
In the 16th century Ferro, Tartaglia, Cardano, . . . , discovered how to solve cubics.

The cubic formula is:

\[ z^3 + pz = q \]

has solutions

\[ z = \sqrt[3]{q^2 + \sqrt{q^4 - 4p^3/27} \pm \sqrt[3]{q^2 + \sqrt{q^4 - 4p^3/27}} - p/3}. \]

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**Cubic formula**

Formulas

\[ z^3 + pz = q \]

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\[ z = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}. \]
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Proof of the cubic formula

Recall the Binomial Thm

\[(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\]

where \(\binom{n}{k} = \frac{n!}{k!(n-k)!}\)

We use Vieta's substitution

\[x = w - \frac{p}{3}w\]
\[q = \left( w^3 - 3w^2p + 3wp^2 + 27p^3 \right) + p(w - \frac{p}{3}w)\]
\[w^3 - p^3 = 27\]

This is equivalent to the quadratic in \(w^3\)
\[0 = w^6 - qw^3 - p^3\]

which has roots
\[w^3 = \frac{q}{2} \pm \sqrt{\frac{q^2}{4} + 4p^3}\]

Substituting back into \(x = w - \frac{p}{3}w\) gives the formula.
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Substituting back into \(x = w - \frac{p}{3w}\) gives the formula.
Powers of complex numbers

Polar form allows us to find $n$-th powers and $n$-th roots of a complex number.

E.g. Find $w = (1 + i)^{18}$.

[Dumb way: multiply out $(1 + i)(1 + i)\ldots(1 + i)$]
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More interestingly, polar forms allow easy computation of roots.

E.g. Solve $z^4 = i$. 
Roots of complex numbers

More interestingly, polar forms allow easy computation of roots.

**E.g.** Solve $z^4 = i$. 
Expressing trigonometric polynomials as polynomials in \( \cos \theta, \sin \theta \)

A trigonometric polynomial is a linear combination of functions of the form \( \cos^n \theta, \sin^n \theta \).

Example:
Use De Moivre's theorem to show \( \cos(3\theta) = 4 \cos^3 \theta - 3 \cos \theta \).

\[
\cos 3\theta = \Re \left( e^{i3\theta} \right) = \Re \left( \cos \theta + i\sin \theta \right)^3 = \cos^3 \theta - 3 \cos \theta \left( 1 - \cos^2 \theta \right) = 4 \cos^3 \theta - 3 \cos \theta.
\]
Expressing trigonometric polynomials as polynomials in $\cos \theta$, $\sin \theta$

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$$\cos 3\theta = \Re(e^{i3\theta}) = \Re(\cos \theta + i\sin \theta)^3 = \Re(\cos^3 \theta + 3i\cos^2 \theta \sin \theta - 3\cos \theta \sin^2 \theta - i\sin^3 \theta) = \cos^3 \theta - 3\cos \theta \sin^2 \theta = 4\cos^3 \theta - 3\cos \theta.$$
Expressing trigonometric polynomials as polynomials in \( \cos \theta, \sin \theta \)

A *trigonometric polynomial* is a linear combination of functions of the form \( \cos n\theta, \sin n\theta \).

**Example:** Use De Moivre’s thm to show \( \cos(3\theta) = 4 \cos^3 \theta − 3 \cos \theta \).

\[
\cos 3\theta = \Re \left( e^{i3\theta} \right)
\]

\[
= \Re (\cos \theta + i \sin \theta)^3
\]

\[
= \Re (\cos^3 \theta + 3i \cos^2 \theta \sin \theta − 3 \cos \theta \sin^2 \theta − i \sin^3 \theta)
\]

\[
= \cos^3 \theta − 3 \cos \theta \sin^2 \theta
\]

\[
= \cos^3 \theta − 3 \cos \theta (1 − \cos^2 \theta)
\]

\[
= 4 \cos^3 \theta − 3 \cos \theta.
\]
Since \( e^{i\theta} = \cos \theta + i\sin \theta \) and \( e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) \), we have

\[
\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.
\]
cos \theta, \sin \theta \text{ in terms of exponentials}

Since

\[ e^{i\theta} = \cos \theta + i \sin \theta \]
\[ e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) \]
\[ = \cos \theta - i \sin \theta \]
\[ = \overline{e^{i\theta}} \]
cos \theta, \sin \theta \text{ in terms of exponentials}

Since

\[ e^{i\theta} = \cos \theta + i \sin \theta \]
\[ e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) \]
\[ = \cos \theta - i \sin \theta \]
\[ = \overline{e^{i\theta}} \]

we have

\[ \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \text{and} \quad i \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2}. \]
Expressing $\cos^n \theta$, $\sin^n \theta$ as trig polynomials

E.g. Prove that

$\sin^4 \theta = \frac{1}{8} \cos 4 \theta - \frac{1}{2} \cos 2 \theta + \frac{3}{8}$.

Thus

$\int \sin^4 \theta \, d\theta = \int \left( e^{i \theta} - e^{-i \theta} \right)^4 \, d\theta = e^{i 4 \theta} - 4 e^{i 2 \theta} + 6 - 4 e^{-i 2 \theta} + e^{-i 4 \theta} \frac{1}{16} = \frac{1}{8} \cos 4 \theta - \frac{1}{2} \cos 2 \theta + \frac{3}{8}$. 
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$$= \frac{e^{4i\theta} - 4e^{2i\theta} + 6 - 4e^{-2i\theta} + e^{-4i\theta}}{16}$$

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Trigonometric sums

Find $\Sigma = \cos \theta + \cos 2\theta + \ldots + \cos n\theta$.

Consider the sum of a geometric progression $S := e^{i\theta} + e^{i2\theta} + \ldots + e^{in\theta} = e^{i(n+1)\theta} - e^{i\theta} - 1$.

Then $\Sigma = \Re S = S + S^2 = \frac{1}{2} (e^{i(n+1)\theta} - e^{i\theta} - 1 + e^{-i(n+1)\theta} - e^{-i\theta} - 1)$.

OR note $S = e^{i\theta} e^{in\theta} - e^{i\theta} - 1 = e^{i\theta} e^{in\theta/2} (e^{in\theta/2} - e^{-in\theta/2}) e^{i\theta/2} (e^{i\theta/2} - e^{-i\theta/2}) = e^{i(n+1)\theta/2} \sin n\theta/2 \sin \theta/2$.

which has real part $\Sigma = \cos ((n+1)\theta/2) \sin n\theta/2 \sin \theta/2$.

Daniel Chan (UNSW)
Q Find $\Sigma = \cos \theta + \cos 2\theta + \ldots + \cos n\theta$. 

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Q Find \( \Sigma = \cos \theta + \cos 2\theta + \ldots + \cos n\theta \).
A Consider the sum of a geometric progression

\[ S := e^{i\theta} + e^{i2\theta} + \ldots + e^{in\theta} = \frac{e^{i(n+1)\theta} - e^{i\theta}}{e^{i\theta} - 1}. \]
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Triangle inequality

The name comes from the following geometric interpretation.
Triangle inequality

\[ |z + w|^2 = (z + w)(z + w) = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} = |z|^2 + z\bar{w} + \bar{z}w + |w|^2 = |z|^2 + 2\text{Re}(z\bar{w}) + |w|^2 \leq |z|^2 + 2|z\bar{w}| + |w|^2 = (|z| + |w|)^2 \]
Triangle inequality

$$|z + w|^2 = (z + w)(\overline{z + w})$$
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\[ |z + w| \leq |z| + |w| \]

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Complex polynomials

Definition

A function $p : \mathbb{C} \to \mathbb{C}$ of the form $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ (with coefficients $a_n, \ldots, a_0 \in \mathbb{C}$) is called a (complex) polynomial.

The degree of $p$, written $\deg(p)$, is the highest power with a non-zero coefficient. If $n$ above is the degree, then $a_n$ is called the leading coefficient.
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The fundamental theorem of algebra

A (complex) root of a polynomial $p$ is any $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.

Theorem (Gauss) Every complex polynomial of degree at least one has a root $\alpha \in \mathbb{C}$.

Note It does not give any formula for the roots (unlike the quadratic and cubic formula).

About the proofs
You will see a proof in your 2nd year complex analysis course.
There is another proof via Galois theory.
Gauss himself gave several proofs, including the following below which requires algebraic topology to make rigorous.
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Factorising polynomials

Let $p, q$ be complex polynomials of degree at least 1. Then $q$ is a factor of $p$ if there is a polynomial $r$ such that $p = qr$. We also say $q$ divides $p$.

For example, $z - 1$ is a factor of $z^3 - 1$ as $z^3 - 1 = (z - 1)(z^2 + z + 1)$.

Theorem (Remainder and Factor)

Let $p$ be a complex polynomial of degree at least one. The remainder on dividing $p$ by $z - \alpha$ is $p(\alpha)$. In particular, $z - \alpha$ is a factor of $p(z)$ if and only if $\alpha$ is a root of $p$.

Proof. Use the long division algorithm for polynomial division to see that $p(z) = (z - \alpha)q(z) + r$ for some polynomial $q(z)$ and remainder $r$ which is constant since its degree must be less than $\text{deg}(z - \alpha)$.

Then $p(\alpha) = r$ which is zero precisely when $\alpha$ is a root or equivalently, $z - \alpha$ is a factor.
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Then $p(\alpha) = r$ which is zero precisely when $\alpha$ is a root or equivalently, $z - \alpha$ is a factor.
Putting the Factor Theorem and the Fundamental Theorem of Algebra together says that if \( p \) is a polynomial of degree \( n \), then there exists \( \alpha_1 \in \mathbb{C} \) such that \( p(z) = (z - \alpha_1)g_1(z) \), where \( g_1(z) \) has degree \( n - 1 \).

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Continuing, you get

**Theorem**

Any degree \( n \) complex polynomial has a factorisation of the form

\[ p(z) = (z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_n)c \]

with \( \alpha_i, c \in \mathbb{C} \).

The terms \((z - \alpha_j)\) are called linear factors of \( p \).

This factorisation is unique up to swapping factors around.

E.g. Factorise \( p(z) = z^3 + z^2 - 2 \) into linear factors.
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Putting the Factor Theorem and the Fundamental Theorem of Algebra together says that if $p$ is a polynomial of degree $n$, then there exists $\alpha_1 \in \mathbb{C}$ such that $p(z) = (z - \alpha_1)g_1(z)$, where $g_1(z)$ has degree $n - 1$.

If $n - 1 \geq 1$ then there exists $\alpha_2 \in \mathbb{C}$ such that $p(z) = (z - \alpha_1)(z - \alpha_2)g_2(z)$. Continuing, you get

**Theorem**

Any degree $n$ complex polynomial has a factorisation of the form

$$p(z) = (z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_n)c$$

with $\alpha_i, c \in \mathbb{C}$. The terms $(z - \alpha_j)$ are called **linear factors** of $p$. This factorisation is unique up to swapping factors around.

**E.g.** Factorise $p(z) = z^3 + z^2 - 2$ into linear factors.
In an example like $p(z) = (z−3)^4(z−i)^2(z+1)$ where the linear factors are not distinct, we say that $(z−3)$ is a factor of multiplicity 4, and that 3 is a root of multiplicity 4. Similarly, $i$ is a root of multiplicity 2 and $-1$ is a root of multiplicity 1.

Q Find all cubic polynomials which have 2 as a root of multiplicity 3.
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Proof of uniqueness of factorisation

Consider two factorisations
\[ p(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n) \]
\[ c = (z - \beta_1)(z - \beta_2) \cdots (z - \beta_n) \]
(1)

We need to show that we can re-order the \( \beta_i \)'s so that \( \alpha_1 = \beta_1, \ldots, \alpha_n = \beta_n, c = d \).

First note \( c = d \) since they are both the leading co-efficient of \( p(z) \).

We argue by induction on \( n \). The case \( n = 0 \) already has been verified so assume \( n > 0 \).

Substitute in \( z = \alpha_1 \) to obtain
\[ 0 = (\alpha_1 - \beta_1)(\alpha_1 - \beta_2) \cdots (\alpha_1 - \beta_n) d \]

One of the RHS factors, say \( \alpha_1 - \beta_i = 0 \).

Swap \( \beta_i, \beta_1 \) so \( \alpha_1 = \beta_1 \).

Dividing (1) by \( z - \alpha_1 \) gives 2 factorisations of \( p(z) \)
\[ z - \alpha_1 = (z - \alpha_2) \cdots (z - \alpha_n) c = (z - \beta_2) \cdots (z - \beta_n) d \]

By induction, we may assume also \( \alpha_2 = \beta_2, \ldots, \alpha_n = \beta_n, c = d \), so we've won.
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Example: factorisation

E.g. Write $p(z) = z^4 + 1$ as a product of linear factors.

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A polynomial is real if the coefficients are real.

Theorem
Suppose that $\alpha$ is a root of a real polynomial $p$. Then $\alpha$ is also a root of $p$.

Proof. Note that in such a case ($z - \alpha$) and ($z - \alpha$) are both factors. If $\alpha \not\in \mathbb{R}$, then unique factorisation $\Rightarrow p(z)$ has a quadratic factor $(z - \alpha)(z - \alpha) = z^2 - (\alpha + \alpha)z + \alpha\alpha = z^2 - (2\text{Re}\alpha)z + |\alpha|^2$, which is a real quadratic.
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Factorising real polynomials

We say a real polynomial $p$ is irreducible over the reals if it can't be factored into a product of two real polynomials of positive degree. E.g. $z^2 - 3z + 2$ is not irreducible but $z^2 + 1$ is.

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Upshot

A real quadratic polynomial is irreducible over $\mathbb{R}$ iff it has non-real roots.

Using the fundamental theorem of algebra and the previous slide (and our old inductive argument) we see

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Any real polynomial can be factored into a product of real linear and real irreducible quadratic polynomials.
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Example: factorisation of a real polynomial

Factorise \( p(z) = z^4 + 1 \) into real irreducible factors.

**Method 1**

Just try your luck with factorisation facts you know

\[ z^4 + 1 = (z^2 + 1)^2 - 2z^2 = (z^2 + \sqrt{2}z + 1)(z^2 - \sqrt{2}z + 1) \]

**OR**

**Method 2**

Recall complex linear factorisation

\[ z^4 + 1 = (z - e^{i\pi/4})(z - e^{-i\pi/4})(z - e^{3i\pi/4})(z - e^{-3i\pi/4}) \]

**Remark**

This is the first instance of common technique in mathematics, to answer a question involving real numbers, first answer it over the complex numbers and deduce your result accordingly.
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