Chapter 3: Complex Numbers

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UNSW

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Philosophical discussion about numbers

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OK. Let's extend our number system by pretending $\sqrt{-1}$ is a number which we'll denote as usual by i, and see what happens.

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Major Question

If we keep playing this game blindly, using our usual rules of arithmetic, will we ever end up proving absurd statements like $1=0$?

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such that the axioms on the following page hold.

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Remark This field is very important in coding theory.

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E.g. Simplify the following expression in a field

$$x(y+z) - yx$$

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2. We often write a for $a + 0i$ and bi for $0 + bi$.

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N.B. $\mathbb{C}$ extends the real number system since complex numbers of form $a + 0i$ add and multiply just like real numbers.

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Eg Simplify $(2 + i)(1 - 3i) - 1 + 3i$

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Q Express $\frac{1+i}{1-i} - \frac{1-i}{1+i}$ in cartesian form.

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E.g. Show that for any $z \in \mathbb{C}$, $(i + 5)z - (i - 5)\overline{z}$ is real.

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Adding complex numbers is by adding real and imaginary parts, i.e. coordinatewise so is represented geometrically by the addition of vectors. Similarly for subtraction.

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Examples: modulus and argument

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E.g. Find the modulus and principal argument of $-1-\sqrt{3}i$.

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E.g. Find the complex number with modulus 3 and argument $\pi/4$.

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Challenge Q What's i^i?

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E.g. Find the exact value of $\mathrm{Arg}\frac{1+i}{1+\sqrt{3}i}$.

Geometry via complex numbers

Q Let $z \in \mathbb{C}$ have $|z| = 1$. Show that $w = \frac{i-z}{i+z}$ is *purely imaginary* in the sense that $\mathrm{Re}\, w = 0$. Interpret the result geometrically.

Square roots of complex numbers

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E.g. Find the complex square roots $\pm z$ of $16 - 30i$

Quadratic formula

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E.g. Solve $z^2 + (1+i)z + (-4+8i) = 0$.

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Even more bizarre fact There's a similar formula for quartics, but can prove no such exists for higher degree (see Galois theory course MATH5725).

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Substituting back into $x = w - \frac{p}{3w}$ gives the formula.

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[Dumb way: multiply out $(1+i)(1+i)\dots(1+i)$]

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E.g. Solve $z^4 = i$.

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Example: Use De Moivre's thm to show $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.

$$
\begin{aligned}
\cos 3\theta &= \mathrm{Re}\left(e^{i3\theta}\right) \\
&= \mathrm{Re}\,(\cos\theta + i\sin\theta)^3 \\
&= \mathrm{Re}\left(\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta\right) \\
&= \cos^3\theta - 3\cos\theta\sin^2\theta \\
&= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\
&= 4\cos^3\theta - 3\cos\theta.
\end{aligned}
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we have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \text{and} \qquad i\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}.$$

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which has real part $\Sigma = \cos\left((n+1)\theta/2\right)\frac{\sin n\theta/2}{\sin\theta/2}$.

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|z+w|^2 &= (z+w)\overline{(z+w)} \\
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&= |z|^2 + z\overline{w} + \overline{z\overline{w}} + |w|^2 \\
&= |z|^2 + 2\mathrm{Re}(z\overline{w}) + |w|^2 \\
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The name comes from the following geometric interpretation.

Complex polynomials

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A function $p : \mathbb{C} \to \mathbb{C}$ of the form

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If n above is the degree, then a_n is called the *leading coefficient*.

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- Gauss himself gave several proofs, including the following below which requires algebraic topology to make rigorous.

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Then $p(\alpha) = r$ which is zero precisely when α is a root or equivalently, $z - \alpha$ is a factor.

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E.g. Factorise $p(z) = z^3 + z^2 - 2$ into linear factors.

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Q Find all cubic polynomials which have 2 as a root of multiplicity 3.

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Consider two factorisations

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By induction, we may assume also $\alpha_2 = \beta_2, \dots, \alpha_n = \beta_n, c = d$, so we've won.

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N.B. Here, the complex roots occur in complex conjugate pairs. This is general phenomena for *real* polynomials.

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Theorem

Any real polynomial can be factored into a product of real linear and real irreducible quadratic polynomials.

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Remark This is the first instance of common technique in mathematics, to answer a question involving real numbers, first answer it over the complex numbers and deduce your result accordingly.