In this chapter, we will answer the following geometric questions:

1. How do you define and then, compute the angle between \( \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \), \( \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \)?

2. How far is the point \( \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \) from the plane \( 2x - y + z = 3 \)?

3. What is the area of the parallelogram in space with sides \( \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \)?

We will in fact, generalise many geometric notions such as angle, to \( \mathbb{R}^n \), by introducing auxiliary gadgets called the dot (or scalar) product and the cross (or vector) product of vectors.
Angles via cosine rule

Let \( a = \overrightarrow{OA}, \ b = \overrightarrow{OB} \in \mathbb{R}^n \) be non-zero. Let’s think about what the angle \( \theta = \angle AOB \) between \( a \) and \( b \) is by considering \( \triangle AOB \) and assuming the cosine rule is valid in \( \mathbb{R}^n \).

### Cosine Rule

\[
|b - a|^2 = |a|^2 + |b|^2 - 2|a||b| \cos \theta.
\]

The formula for the length of a vector gives

\[
\sum_i (b_i - a_i)^2 = \sum_i a_i^2 + \sum_i b_i^2 - 2|a||b| \cos \theta
\]

Cancelling gives \(-2|a||b| \cos \theta = \sum_i -2a_i b_i\) so

\[
\cos \theta = \frac{\sum_i a_i b_i}{|a||b|}
\]

We can solve for \( \theta \) as long as RHS lies in \([-1, 1]\) (see Cauchy-Schwarz thm later).
Remarks concerning above thought experiment

- The distance function \( d(a, b) = \sqrt{\sum_i (b_i - a_i)^2} \) gives you information not just about lengths, but angles too.
- It’s actually better not to base our theory on this function, but on the numerator expression for \( \cos \theta \), i.e. \( \sum_i a_i b_i \).

**Definition**

For \( a, b \in \mathbb{R}^n \), we define the *dot* or *scalar product* of \( a, b \) to be

\[
a \cdot b = \sum_i a_i b_i.
\]

We prove later

**Theorem (Cauchy-Schwarz)**

\[
-|a||b| \leq a \cdot b \leq |a||b|.
\]
Cauchy-Schwarz thm \implies we may now define

**Definition**

The *angle* between non-zero vectors $\mathbf{a} = \overrightarrow{OA}, \mathbf{b} = \overrightarrow{OB} \in \mathbb{R}^n$ is

$$\theta = \cos^{-1} \left( \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| ||\mathbf{b}||} \right)$$

so $\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \cos \theta$.

Of course, this recovers the old definition in $\mathbb{R}^2, \mathbb{R}^3$ since the cosine rule is fine there.

**Example.** Let $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix}$.

What is the angle between $\mathbf{a}$ and $\mathbf{b}$?
Properties of the dot product

Proposition
Suppose $a, b, c \in \mathbb{R}^n$. Then

1. $a \cdot b = b \cdot a$.
2. $a \cdot (\lambda b) = (\lambda a) \cdot b = \lambda (a \cdot b)$.
3. $a \cdot (b + c) = a \cdot b + a \cdot c$.
4. $a \cdot a = |a|^2 \geq 0$.

Note the last, means that the dot product gives the length function and thus angles can be written out in terms of dot products alone too.

To prove these, just write things out using the definition!
Proof of the Cauchy-Schwarz thm

We prove

Theorem (Cauchy-Schwarz)

\[-|a||b| \leq a \cdot b \leq |a||b|.\]

Proof. The inequality holds when \(b = 0\) so we assume \(b \neq 0\). Consider the real function of (the real variable) \(\lambda\)

\[q(\lambda) = |a - \lambda b|^2 \geq 0.\]

\[q(\lambda) = (a - \lambda b) \cdot (a - \lambda b) = |a|^2 - 2\lambda a \cdot b + \lambda^2 |b|^2.\]  

The discriminant of this quadratic function of \(\lambda\) must be non-positive, hence

\[0 \geq (-2a \cdot b)^2 - 4|a|^2|b|^2 = 4(a \cdot b)^2 - 4|a|^2|b|^2 \implies (a \cdot b)^2 \leq |a|^2|b|^2\]
Orthogonality

Definition

Two vectors \( a, b \in \mathbb{R}^n \) are said to be orthogonal if \( a \cdot b = 0 \) i.e. the angle between them is 0.

\[ a = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \]

are orthogonal.

Theorem (Pythagoras)

If \( a, b \in \mathbb{R}^n \) are orthogonal then

\[ |a + b|^2 = |a|^2 + |b|^2. \]

Proof.
Question

Show that the altitudes of \( \triangle ABC \) are concurrent.

A Let \( P \) be the intersection of the altitudes through \( A \) and \( B \). It suffices to show that \( PC \) is an altitude too.

We may pick \( C \) to be the origin \( O \) & let \( p, a, b, 0 \) be the position vectors of \( P, A, B, C \).
Orthonormal sets of vectors

Definition

The vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n \) form an \textit{orthogonal} set if they are mutually orthogonal. If furthermore, they all have length 1, we say they are \textit{orthonormal}.

Equivalently, \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) are orthonormal if

\[
\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij} := \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{if } i \neq j.
\end{cases}
\]

[\( \delta_{ij} \) is called the Kronecker delta symbol.]

The standard basis vector \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) obviously are orthonormal.

\textbf{E.g.} The vectors \( \mathbf{v}_1 = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \), \( \mathbf{v}_2 = \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \) are orthonormal.
Linear combinations of orthonormal vectors

E.g. Express \( \begin{pmatrix} 2 \\ -1 \end{pmatrix} \) as a linear combination of \( \mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \) (if possible).

A Possible because
We can prescribe a plane $P$ by giving a point $a$ on the plane, and its orientation which is usually done by giving 2 non-parallel vectors giving “directions”. In $\mathbb{R}^3$ the orientation, can also be given by a normal vector $n$, i.e. so $n$ is perpendicular to every vector parallel to the plane.

The plane $P$ is the set of all point $x$ such that

$$n \cdot (x - a) = 0.$$  \hspace{1cm} \text{(PN)}

or equivalently

$$n \cdot x = n \cdot a \quad \text{(3)}$$

These are called the point-normal form of the plane.

We can re-write (3) in Cartesian form

$$n_1x_1 + n_2x_2 + n_3x_3 = b$$

where $b$ is the constant $n \cdot a$. 

Example: point-normal form

**E.g.** Find the Cartesian form for the plane in $\mathbb{R}^3$ with normal \[
\begin{pmatrix}
1 \\
-1 \\
2
\end{pmatrix}
\] passing through \[
\begin{pmatrix}
1 \\
1 \\
3
\end{pmatrix}.
\]

**E.g.** Find a normal to the plane $P_1 : 3x - 2y + 5z = 7$.

**Challenge Q** What’s the angle between $P_1$ and $P_2 : x + y + z = 9$?
Let \( b \in \mathbb{R}^n \) and \( v \in \mathbb{R}^n - \mathbf{0} \). Informally, the *projection of \( b \) onto \( v \)* is obtained by dropping a perpendicular from the head of \( b \) to the line through \( O \) in the direction \( v \).

This projection has form \( c = \lambda v \) for some \( \lambda \in \mathbb{R} \).

To determine \( \lambda \), we use trig to see \( |c| = |b| \cos \theta \) where \( \theta = \) angle between \( b, v \).

Hence
\[
c = |b| \cos \theta \frac{v}{|v|} = \frac{b \cdot v}{|v|} \frac{v}{|v|}.
\]
The above suggests,

**Definition**

For $b \in \mathbb{R}^n$ and $v \in \mathbb{R}^n - 0$, the *projection of $b$ onto $v$* is

$$\text{proj}_v b = \left( \frac{b \cdot v}{|v|^2} \right) v.$$ 

**E.g.** Find $\text{proj}_v b$ when $v = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$. 
Properties of the projection

Our definition agrees with the procedure of dropping a perpendicular by

**Proposition**

\[ \text{proj}_v b \] is the unique vector of the form \( \lambda v \) such that \( b - \lambda v \) is orthogonal to \( v \).

**Proof.** \( 0 = (b - \lambda v) \cdot v = b \cdot v - \lambda |v|^2 \) has unique soln
\[ \lambda = \frac{b \cdot v}{|v|^2}. \]

**Proposition**

\[ \text{proj}_v b \] is the unique point on the line \( x = \lambda v, \lambda \in \mathbb{R} \), closest to \( b \).

**Proof.** It’s best to see this with a picture and use Pythagoras.
Distance between a point and a line

**E.g.** Find the point on the line

\[ x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad \lambda \in \mathbb{R}, \]

closest to \( b = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}. \)

Find this distance from \( b \) to the line.
**E.g.** Find the distance between the plane $P : x_1 + x_2 + x_3 = 0$ and $b = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$.

**A** If $c$ gives the point on $P$ which is closest to $b$, then our argument using Pythagoras thm says that we should have $b - c$ is orthogonal to $P$ i.e. $b - c$ is parallel to
Determinants

We will look at determinants more fully in chapter 5. Here’s what we need for now. Below $a_i, b_i, e_i$ are real (and later complex) scalars.

**Definition**

We define the $2 \times 2$ determinant by

$$|a_1 \ a_2| \ b_1 \ b_2 = a_1 b_2 - a_2 b_1.$$ 

**E.g.**

The $3 \times 3$ determinant is defined by

$$|e_1 \ e_2 \ e_3| \ a_1 \ a_2 \ a_3 | = e_1 |a_2 \ a_3| - e_2 |a_1 \ a_3| + e_3 |a_1 \ a_2|.$$
Proposition

If you swap two rows of a determinant, it changes sign e.g.

\[
\begin{vmatrix}
  e_1 & e_2 & e_3 \\
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
\end{vmatrix}
= -
\begin{vmatrix}
  e_1 & e_2 & e_3 \\
  b_1 & b_2 & b_3 \\
  a_1 & a_2 & a_3 \\
\end{vmatrix}
\]

In particular, the determinant is 0 if 2 rows are the same (swapping them both negates and keeps them the same).

Proof For 2 × 2-matrices

\[
\begin{vmatrix}
  b_1 & b_2 \\
  a_1 & a_2 \\
\end{vmatrix}
= b_1a_2 - b_2a_1 = -
\begin{vmatrix}
  a_1 & a_2 \\
  b_1 & b_2 \\
\end{vmatrix}
\]

3 × 3 case is harder
Cross product

For \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \), the cross or vector product of \( \mathbf{a} \) and \( \mathbf{b} \) is the vector

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix}
\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{vmatrix}
\]

\[
= \mathbf{e}_1 \begin{vmatrix}
a_2 & a_3 \\
b_2 & b_3
\end{vmatrix} - \mathbf{e}_2 \begin{vmatrix}
a_1 & a_3 \\
b_1 & b_3
\end{vmatrix} + \mathbf{e}_3 \begin{vmatrix}
a_1 & a_2 \\
b_1 & b_2
\end{vmatrix}
\]

N.B. The second term is not really well-defined, but is a useful mnemonic.

E.g. Find \( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} \).

N.B. There are higher dimensional versions of the cross product, but they are much more complicated and not as useful.
Proposition

For \( a, b, c \in \mathbb{R}^3 \) and \( \lambda \in \mathbb{R} \) we have

1. \( a \times a = 0 \) (by row swapping & determinants).
2. \( \times \) is distributive:
   \[
   a \times (b + c) = (a \times b) + (a \times c)
   \]
   \[
   (a + b) \times c = (a \times c) + (b \times c).
   \]
3. \( (\lambda a) \times b = \lambda (a \times b) = a \times (\lambda b) \)

**Proof.** Just expand both sides with the defn! Alternately, wait until we’ve looked at more properties of determinants in chapter 5.

**Warning**

- \( a \times b = -b \times a \) (by row swapping of determinants) so \( \times \) is not commutative!
- The cross product is NOT associative!
The magnitude of $a \times b$

**Theorem**

Suppose that $a, b \in \mathbb{R}^3$ are nonzero, with angle $\theta$ between them. Then 

$$|a \times b| = |a||b| \sin \theta.$$ 

**Proof.** Remember that $\theta \in [0, \pi]$ in $\mathbb{R}^3$ so $\sin \theta \geq 0$. Thus, it is enough to show that 

$$|a \times b|^2 = |a|^2 |b|^2 \sin^2 \theta.$$ 

Remembering the definition of $\theta$:

$$\sin^2 \theta = 1 - \cos^2 \left( \cos^{-1} \left( \frac{a \cdot b}{|a||b|} \right) \right) = 1 - \left( \frac{a \cdot b}{|a||b|} \right)^2$$

and so 

$$|a|^2 |b|^2 \sin^2 \theta = |a|^2 |b|^2 - (a \cdot b)^2.$$ 

Now get MAPLE to expand out 

$$|a \times b|^2 - (|a|^2 |b|^2 - (a \cdot b)^2)$$

in terms of the coordinates of $a, b$ & check it is zero!
Areas of parallelograms via cross product

**Proposition**

The area of the parallelogram with sides \( a, b \) is \(|a \times b|\).

**Proof** The area of the parallelogram is

\[
A = \text{base} \times \text{perp height} = |a||b|\sin \theta.
\]

Thm previous slide gives the result.

**E.g.** Find the area of the parallelogram with vertices at \((1, 1), (4, 2), (2, 3)\) and \((5, 4)\).
Scalar triple product

Let \( a, b, e \in \mathbb{R}^3 \).

**Proposition-Definition**

The scalar

\[
e \cdot (a \times b) = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\]

It is called the *scalar triple product of* \( e, a, b \).

**Proof** If \( c = a \times b \) then

\[
e \cdot (a \times b) = e_1 c_1 + e_2 c_2 + e_3 c_3 = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.
\]

**E.g.** Find \( e_1 \cdot (e_2 \times e_3) \).
Direction of $a \times b$

Row swapping of determinants give the following

**Proposition**

1. $a \cdot (b \times c) = -a \cdot (c \times b) = c \cdot (a \times b)$.
2. $a \cdot (a \times b) = 0 = a \cdot (b \times b)$.

Part 2) gives

**Proposition**

The vector $a \times b$ is orthogonal to both $a$ and $b$.

**N.B** This proposition and our formula for $|a \times b|$ determines $a \times b$ up to a choice of two vectors.

The choice of which one is given by the *right hand rule*. 

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Chapter 2: Vector Geometry
Term 1 2020  26 / 31
There are many geometric problems in $\mathbb{R}^3$ where one needs to find a vector which is orthogonal to two given vectors.

**Example.** Find a point-normal, and hence a Cartesian form for the plane

$$\mathbf{x} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$  

A point on the plane is $\mathbf{a} =$

A vector normal to the plane is $\mathbf{n} =$

Thus a point-normal form is
**Problem.** What is the shortest distance between the two lines $L_1, L_2 \subset \mathbb{R}^3$?

Our argument via Pythagoras thm shows that the shortest line segment joining the two lines needs to be orthogonal to both the lines, that is orthogonal to the two direction vectors.

**Proposition**

Let $\mathbf{n}$ be orthogonal to both lines. Then the shortest distance between the lines equals the length of the vector $\text{proj}_n(\mathbf{a}_1 - \mathbf{a}_2)$, where $\mathbf{a}_j$ is any point on $L_j$.

Why?
Example: distance between lines

**Problem.** What is the shortest distance between the two lines

\[ L_1 : \quad x = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R} \quad L_2 : \quad x = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}, \quad \mu \in \mathbb{R} \]

**A** Here we can take \( \mathbf{n} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ -8 \end{pmatrix} \), and \( \mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \),

so \( \mathbf{a}_1 - \mathbf{a}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \).

The shortest distance between the lines is

\[
|\text{proj}_n(\mathbf{a}_1 - \mathbf{a}_2)| = \left| \frac{\mathbf{n} \cdot (\mathbf{a}_1 - \mathbf{a}_2)}{|\mathbf{n}|^2} \mathbf{n} \right|
\]

\[
= \left| \frac{\mathbf{n} \cdot (\mathbf{a}_1 - \mathbf{a}_2)}{|\mathbf{n}|} \right|
\]
Volumes of parallelepipeds

A *parallelepiped* is a 3-dim version of a parallelogram.

Consider a parallelepiped $P$ with edges $a = \overrightarrow{OA}$, $b = \overrightarrow{OB}$, $c = \overrightarrow{OC}$. Let $n = b \times c$.

If the base of $P$ is the parallelogram sides $b$, $c$, then the perpendicular height $P$ is the length of the projection of $a$ onto $n$. Hence

$$\text{Volume of } P = \text{area base} \times \text{perp. height}$$

$$= |b \times c| |\text{proj}_n a|$$

$$= |b \times c| \left| \frac{a \cdot (b \times c)}{|b \times c|} \right|$$

$$= |a \cdot (b \times c)|.$$
Example: volume of parallelepiped

Example. Find the volume of a parallelepiped with vertices at

\[
\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}
\]

adjacent to the vertex \( \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \).