Chapter 2: Vector Geometry

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Goals of this chapter

In this chapter, we will answer the following geometric questions involving "measurement" **eg** How far is the point $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ from the plane $2x - y + z = 3$?

Key To do so, we introduce two marvellous gadgets called the *dot (or scalar) product* and the *cross (or vector) product* of vectors.

Definition

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, we define the *dot* or *scalar product* of $\mathbf{a}, \mathbf{b}$ to be

$$\mathbf{a} \cdot \mathbf{b} = \sum_i a_i b_i .$$

Eg $\mathbf{a} \cdot \mathbf{a} =$

Angles

Assume for now, the Cauchy-Schwarz thm

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

Definition

The *angle* between non-zero vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right) \quad \text{so} \quad \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta.$$

In particular, $\mathbf{a}, \mathbf{b}$ are *orthogonal* if

See later that angles between geom vectors same as angles between their coordinate vectors.

Example. What is the angle between $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix}$.

Properties of the dot product

Proposition

Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$. Then

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
2. $\mathbf{a} \cdot (\lambda \mathbf{b}) = (\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$.
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.
4. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \geq 0$.

Note the last, means that the dot product gives the length function and thus angles can be written out in terms of dot products alone too.

To prove these, just write things out using the definition!

Proof of the Cauchy-Schwarz thm

We prove

Theorem (Cauchy-Schwarz)

$$-|\mathbf{a}||\mathbf{b}| \leq \mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}||\mathbf{b}|.$$

Proof. The inequality holds when $\mathbf{b} = \mathbf{0}$ so we assume $\mathbf{b} \neq \mathbf{0}$.
Consider the real function of (the real variable) λ

$$q(\lambda) = |\mathbf{a} - \lambda\mathbf{b}|^2 \geq 0.$$

$$\begin{aligned} q(\lambda) &= (\mathbf{a} - \lambda\mathbf{b}) \cdot (\mathbf{a} - \lambda\mathbf{b}) && (1) \\ &= |\mathbf{a}|^2 - 2\lambda\mathbf{a} \cdot \mathbf{b} + \lambda^2|\mathbf{b}|^2. && (2) \end{aligned}$$

The discriminant of this quadratic function of λ must be non-positive, hence

$$0 \geq (-2\mathbf{a} \cdot \mathbf{b})^2 - 4|\mathbf{a}|^2|\mathbf{b}|^2 = 4(\mathbf{a} \cdot \mathbf{b})^2 - 4|\mathbf{a}|^2|\mathbf{b}|^2 \Longrightarrow (\mathbf{a} \cdot \mathbf{b})^2 \leq |\mathbf{a}|^2|\mathbf{b}|^2$$

Cosine rule and Pythagoras theorem

Theorem

$$|\mathbf{b}-\mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta.$$

Hence if $\mathbf{a} \perp \mathbf{b}$

Proof.

Since the cosine rule holds in space, this identifies the classical defn of angle for geom vectors & abstract defn via dot product.

Application: Orthocentre

Question

Show that the altitudes of $\triangle ABC$ are concurrent.

A Let P be the intersection of the altitudes through A and B. It suffices to show that PC is an altitude too.
We may pick C to be the origin O & let $\mathbf{p}, \mathbf{a}, \mathbf{b}, \mathbf{0}$ be the position vectors of P, A, B, C.

Orthonormal sets of vectors

Definition

The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ form an *orthogonal* set if they are mutually orthogonal. If furthermore, they all have length 1, we say they are *orthonormal* i.e.

$$\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

E.g. Show $\mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ are o/n & express $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ as a linear combination of them.

Point-normal forms for planes in $\mathbb{R}^3$

We can prescribe a plane P by giving a point $\mathbf{a}$ on the plane, and its orientation which is usually done by giving 2 non-parallel vectors giving "directions".
In $\mathbb{R}^3$ the orientation, can also be given by a *normal vector* $\mathbf{n}$, i.e. so $\mathbf{n}$ is perpendicular to every vector parallel to the plane.

The plane P is the set of all point $\mathbf{x}$ such that

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0. \qquad \text{(PN)}$$

or equivalently

$$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{a} \tag{3}$$

These are called the *point-normal form* of the plane.
We can re-write (3) in Cartesian form

$$n_1 x_1 + n_2 x_2 + n_3 x_3 = b$$

where b is the constant $\mathbf{n} \cdot \mathbf{a}$.

Example: point-normal form

E.g. Find the Cartesian form for the plane in $\mathbb{R}^3$ with normal $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ passing through $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$.

E.g. Find a normal to the plane $P_1 : 3x - 2y + 5z = 7$.

Challenge Q What's the angle between P_1 and $P_2 : x + y + z = 9$?

Projection

Definition

For $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n - \mathbf{0}$, the *projection of* $\mathbf{b}$ *onto* $\mathbf{v}$ is

$$\mathrm{proj}_{\mathbf{v}}\mathbf{b} = \left(\frac{\mathbf{b} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}.$$

It's the unique vector of the form $\lambda\mathbf{v}$ such that $\mathbf{b} - \lambda\mathbf{v}$ is orthogonal to $\mathbf{v}$.

Proof. $0 = (\mathbf{b} - \lambda\mathbf{v}) \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v} - \lambda|\mathbf{v}|^2$ has unique soln

Proposition

$\mathrm{proj}_{\mathbf{v}}\mathbf{b}$ is the unique point on the line $\mathbf{x} = \lambda\mathbf{v}, \lambda \in \mathbb{R}$, closest to $\mathbf{b}$.

Proof. It's best to see this with a picture and use Pythagoras.

Distance between a point and a line

E.g. Find the point on the line

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \qquad \lambda \in \mathbb{R},$$

closest to $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

Find this distance from **b** to the line.

Distance from a point to a plane

E.g. Find the distance between the plane $P : x_1 + x_2 + x_3 = 0$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$.

A If $\mathbf{c}$ gives the point on P which is closest to $\mathbf{b}$, then our argument using Pythagoras thm says that we should have $\mathbf{b} - \mathbf{c}$ is orthogonal to P i.e. $\mathbf{b} - \mathbf{c}$ is parallel to

Determinants

We will look at determinants more fully in chapter 5. Here's what we need for now. Below a_i, b_i, e_i are real (and later complex) scalars.

Definition

We define the 2×2 *determinant* by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

E.g.

The 3×3 determinant is defined by

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = e_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - e_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + e_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Determinants satisfy many beautiful formulae such as

Formula (row scaling) $\begin{vmatrix} \lambda a_1 & \lambda a_2 \\ b_1 & b_2 \end{vmatrix} =$

Cross product

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, the *cross* or *vector product* of **a** and **b** is the vector

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \mathbf{e}_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{aligned}$$

N.B. The second term is not really well-defined, but is a useful mnemonic.

E.g. Find $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$.

Arithmetic properties of the cross product

Proposition

For $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$ we have

1. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
2. $\times$ is distributive:
$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$$
$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}).$$
3. $(\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\lambda \mathbf{b})$ by row scaling formula

Proof. Just expand both sides with the defn! Alternately, wait until we've looked at more properties of determinants in chapter 5.

Warning

- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ so $\times$ is not commutative!
- The cross product is NOT associative!

The magnitude of $\mathbf{a} \times \mathbf{b}$

Theorem

Suppose $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ have angle θ between them. Then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$.

Proof. Note $\theta \in [0, \pi]$ in $\mathbb{R}^3$ so $\sin\theta \geq 0$ $\quad\therefore$ suff show

$$|\mathbf{a} \times \mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2\theta = 0 \qquad \dots(*)$$

Can assume $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$, so using defn of θ:

$$\sin^2\theta = 1 - \cos^2\left(\cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)\right) = 1 - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)^2$$

and so

$$|\mathbf{a}|^2|\mathbf{b}|^2 \sin^2\theta = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2.$$

Now

$$LHS(*) = |\mathbf{a} \times \mathbf{b}|^2 - (|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2) =$$

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 - \sum_i a_i^2 \sum_i b_i^2 + (\sum_i a_i b_i)^2 =$$

Areas of parallelograms via cross product

Proposition

The area of the parallelogram with sides $\mathbf{a}, \mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$.

Proof The area of the parallelogram is

$$A = \text{base x perp height} = |\mathbf{a}||\mathbf{b}| \sin\theta.$$

Thm previous slide gives the result.

E.g. Find the area of the parallelogram with vertices at $(1,1)$, $(4,2)$, $(2,3)$ and $(5,4)$.

Scalar triple product

Let $\mathbf{a}, \mathbf{b}, \mathbf{e} \in \mathbb{R}^3$.

Proposition-Definition

The scalar

$$\mathbf{e} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

It is called the *scalar triple product of* $\mathbf{e}, \mathbf{a}, \mathbf{b}$.

Proof If $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ then

$$\mathbf{e} \cdot (\mathbf{a} \times \mathbf{b}) = e_1 c_1 + e_2 c_2 + e_3 c_3 = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Direction of $\mathbf{a} \times \mathbf{b}$

E.g. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = a_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} =$

Proposition

The vector $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

N.B This proposition and our formula for $|\mathbf{a} \times \mathbf{b}|$ determines $\mathbf{a} \times \mathbf{b}$ up to a choice of two vectors.

The choice of which one is given by the *right hand rule*.

Application: Parametric to cartesian form via point-normal

There are many geometric problems in $\mathbb{R}^3$ where one needs to find a vector which is orthogonal to two given vectors.

Example. Find a point-normal, and hence a Cartesian form for the plane

$$\mathbf{x} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$

A A point on the plane is $\mathbf{a} =$

A vector normal to the plane is $\mathbf{n} =$
Thus a point-normal form is

Application: distance between lines

Problem. What is the shortest distance between the two lines $L_1, L_2 \subset \mathbb{R}^3$?

Our argument via Pythagoras thm shows that the shortest line segment joining the two lines needs to be orthogonal to both the lines, that is orthogonal to the two direction vectors.

Proposition

Let $\mathbf{n}$ be orthogonal to both lines. Then the shortest distance between the lines equals the length of the vector $\mathrm{proj}_{\mathbf{n}}(\mathbf{a}_1 - \mathbf{a}_2)$, where $\mathbf{a}_j$ is any point on L_j.

Why?

Example: distance between lines

Problem. What is the shortest distance between the two lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda \begin{pmatrix}2\\2\\1\end{pmatrix}, \ \lambda \in \mathbb{R} \qquad L_2 : \quad \mathbf{x} = \begin{pmatrix}2\\-1\\1\end{pmatrix} + \mu \begin{pmatrix}4\\0\\3\end{pmatrix}, \ \mu \in \mathbb{R}?$$

A Here we can take $\mathbf{n} = \begin{pmatrix}2\\2\\1\end{pmatrix} \times \begin{pmatrix}4\\0\\3\end{pmatrix} = \begin{pmatrix}6\\-2\\-8\end{pmatrix}$, and $\mathbf{a}_1 = \begin{pmatrix}1\\0\\2\end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix}2\\-1\\1\end{pmatrix}$,

so $\mathbf{a}_1 - \mathbf{a}_2 = \begin{pmatrix}-1\\1\\1\end{pmatrix}$.

The shortest distance between the lines is

$$\begin{aligned} |\mathrm{proj}_{\mathbf{n}}(\mathbf{a}_1 - \mathbf{a}_2)| &= \left| \frac{\mathbf{n} \cdot (\mathbf{a}_1 - \mathbf{a}_2)}{|\mathbf{n}|^2} \mathbf{n} \right| \\ &= \left| \frac{\mathbf{n} \cdot (\mathbf{a}_1 - \mathbf{a}_2)}{|\mathbf{n}|} \right| \\ &= \end{aligned}$$

Volumes of parallelepipeds

A *parallelepiped* is a 3-dim version of a parallelogram.

Consider a parallelepiped P with edges $\mathbf{a} = \overrightarrow{OA}, \mathbf{b} = \overrightarrow{OB}, \mathbf{c} = \overrightarrow{OC}$. Let $\mathbf{n} = \mathbf{b} \times \mathbf{c}$.

If the base of P is the parallelogram sides $\mathbf{b}, \mathbf{c}$, then the perpendicular height P is the length of the projection of $\mathbf{a}$ onto $\mathbf{n}$. Hence

$$\begin{aligned}
\text{Volume of } P &= \text{area base} \times \text{perp. height} \\
&= |\mathbf{b} \times \mathbf{c}| \, |\mathrm{proj}_{\mathbf{n}} \mathbf{a}| \\
&= |\mathbf{b} \times \mathbf{c}| \left| \frac{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}{|\mathbf{b} \times \mathbf{c}|} \right| \\
&= |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.
\end{aligned}$$

Example: volume of parallelepiped

Example. Find the volume of a parallelepiped with vertices at

$$\begin{pmatrix}0\\0\\2\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}4\\0\\3\end{pmatrix}$$

adjacent to the vertex $\begin{pmatrix}0\\-1\\2\end{pmatrix}$.

A