# Chapter 2: Vector Geometry

Daniel Chan

UNSW

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$$\theta = \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right) \quad \text{so} \quad \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta.$$

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**Example.** What is the angle between $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix}$.

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To prove these, just write things out using the definition!

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We prove

## Theorem (Cauchy-Schwarz)

$$-|\mathbf{a}||\mathbf{b}| \le \mathbf{a} \cdot \mathbf{b} \le |\mathbf{a}||\mathbf{b}|.$$

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$$q(\lambda) = |\mathbf{a} - \lambda\mathbf{b}|^2 \geq 0.$$

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$$\begin{aligned} q(\lambda) &= (\mathbf{a} - \lambda \mathbf{b}) \cdot (\mathbf{a} - \lambda \mathbf{b}) &\quad (1) \\ &= |\mathbf{a}|^2 - 2\lambda \mathbf{a} \cdot \mathbf{b} + \lambda^2 |\mathbf{b}|^2. &\quad (2) \end{aligned}$$

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The discriminant of this quadratic function of $\lambda$ must be non-positive, hence

$$0 \geq (-2\mathbf{a} \cdot \mathbf{b})^2 - 4|\mathbf{a}|^2|\mathbf{b}|^2 = 4(\mathbf{a} \cdot \mathbf{b})^2 - 4|\mathbf{a}|^2|\mathbf{b}|^2 \Longrightarrow (\mathbf{a} \cdot \mathbf{b})^2 \leq |\mathbf{a}|^2|\mathbf{b}|^2$$

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**Theorem**

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Since the cosine rule holds in space, this identifies the classical defn of angle for geom vectors & abstract defn via dot product.

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We may pick $C$ to be the origin $O$ & let $\mathbf{p}, \mathbf{a}, \mathbf{b}, \mathbf{0}$ be the position vectors of $P, A, B, C$.

# Orthonormal sets of vectors

## Definition

The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ form an *orthogonal* set if they are mutually orthogonal. If furthermore, they all have length 1, we say they are *orthonormal* i.e.

$$\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

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**E.g.** Show $\mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ are o/n & express $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ as a linear combination of them.

# Point-normal forms for planes in $\mathbb{R}^3$

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We can re-write (3) in Cartesian form

$$n_1 x_1 + n_2 x_2 + n_3 x_3 = b$$

where $b$ is the constant $\mathbf{n} \cdot \mathbf{a}$.

# Example: point-normal form

**E.g.** Find the Cartesian form for the plane in $\mathbb{R}^3$ with normal $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ passing through $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$.

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**Challenge Q** What's the angle between $P_1$ and $P_2 : x + y + z = 9$?

# Projection

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It's the unique vector of the form $\lambda\mathbf{v}$ such that $\mathbf{b} - \lambda\mathbf{v}$ is orthogonal to $\mathbf{v}$.

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**Proof.** It's best to see this with a picture and use Pythagoras.

# Distance between a point and a line

**E.g.** Find the point on the line

$$\mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda \begin{pmatrix}2\\1\\-2\end{pmatrix}, \qquad \lambda \in \mathbb{R},$$

closest to $\mathbf{b} = \begin{pmatrix}3\\2\\1\end{pmatrix}$.

Find this distance from $\mathbf{b}$ to the line.

**E.g.** Find the distance between the plane $P : x_1 + x_2 + x_3 = 0$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$.

# Distance from a point to a plane

**E.g.** Find the distance between the plane $P : x_1 + x_2 + x_3 = 0$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$.

**A** If $\mathbf{c}$ gives the point on $P$ which is closest to $\mathbf{b}$, then our argument using Pythagoras thm says that we should have $\mathbf{b} - \mathbf{c}$ is orthogonal to $P$ i.e. $\mathbf{b} - \mathbf{c}$ is parallel to

# Determinants

We will look at determinants more fully in chapter 5. Here's what we need for now. Below $a_i, b_i, e_i$ are real (and later complex) scalars.

## Definition

We define the $2 \times 2$ *determinant* by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

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The $3 \times 3$ determinant is defined by

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Determinants satisfy many beautiful formulae such as

**Formula** (row scaling) $\begin{vmatrix} \lambda a_1 & \lambda a_2 \\ b_1 & b_2 \end{vmatrix} =$

# Cross product

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, the *cross* or *vector product* of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \mathbf{e}_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

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$$= \mathbf{e}_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

N.B. The second term is not really well-defined, but is a useful mnemonic.

# Cross product

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, the *cross* or *vector product* of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \mathbf{e}_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{aligned}$$

N.B. The second term is not really well-defined, but is a useful mnemonic.

**E.g.** Find $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$.

# Arithmetic properties of the cross product

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**Proof.** Just expand both sides with the defn! Alternately, wait until we've looked at more properties of determinants in chapter 5.

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- The cross product is NOT associative!

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Now

$$LHS(*) = |\mathbf{a} \times \mathbf{b}|^2 - (|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2) =$$

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 - \sum_i a_i^2 \sum_i b_i^2 + (\sum_i a_i b_i)^2 =$$

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**E.g.** Find the area of the parallelogram with vertices at $(1,1)$, $(4,2)$, $(2,3)$ and $(5,4)$.

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## Proposition-Definition

The scalar

$$\mathbf{e} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

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# Direction of $\mathbf{a} \times \mathbf{b}$

**E.g.** $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = a_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} =$

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The choice of which one is given by the *right hand rule*.

# Application: Parametric to cartesian form via point-normal

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**Example.** Find a point-normal, and hence a Cartesian form for the plane

$$\mathbf{x} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}\,.$$

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Let $\mathbf{n}$ be orthogonal to both lines. Then the shortest distance between the lines equals the length of the vector $\mathrm{proj}_{\mathbf{n}}(\mathbf{a}_1 - \mathbf{a}_2)$, where $\mathbf{a}_j$ is any point on $L_j$.

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**Why?**

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**A** Here we can take $\mathbf{n} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ -8 \end{pmatrix}$, and $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, so $\mathbf{a}_1 - \mathbf{a}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

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The shortest distance between the lines is

$$\begin{aligned} |\mathrm{proj}_{\mathbf{n}}(\mathbf{a}_1 - \mathbf{a}_2)| &= \left| \frac{\mathbf{n} \cdot (\mathbf{a}_1 - \mathbf{a}_2)}{|\mathbf{n}|^2} \mathbf{n} \right| \\ &= \left| \frac{\mathbf{n} \cdot (\mathbf{a}_1 - \mathbf{a}_2)}{|\mathbf{n}|} \right| \\ &= \end{aligned}$$

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$$\begin{aligned} \text{Volume of } P &= \text{area base} \times \text{perp. height} \\ &= |\mathbf{b} \times \mathbf{c}|\, |\mathrm{proj}_{\mathbf{n}} \mathbf{a}| \\ &= |\mathbf{b} \times \mathbf{c}| \left| \frac{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}{|\mathbf{b} \times \mathbf{c}|} \right| \\ &= |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|. \end{aligned}$$

# Example: volume of parallelepiped

**Example.** Find the volume of a parallelepiped with vertices at

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