# Chapter 1: Introduction to Vectors

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**Remark** This question illustrates the two fundamental operations of vectors that we'll be studying in this chapter.

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**Vector scalar multiplication** Given a scalar $\lambda \in \mathbb{R}$

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**Why are these true?**

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**Q** Show that the diagonals of a parallelogram bisect each other.

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$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} := \begin{pmatrix} a_1 + \phantom{b_1} \\ a_2 + \phantom{b_2} \\ \vdots \\ a_n + \phantom{b_n} \end{pmatrix}, \qquad \lambda \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} := \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{pmatrix}.$$

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**Rem** These operations satisfy the commutative, associative & distributive laws we saw for geometric vectors.

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**Rem** In fancy language, there's a 1-1 correspondence between $\mathbb{R}^2$ and $V$.

# Co-ordinate arithmetic reflects vector arithmetic

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ so coords are $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

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**3-dim version** Sim if you pick mutually orthogonal, unit length vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in space, $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ are the coordinates of $a_1\mathbf{i} + a_2\mathbf{j} + a_2\mathbf{k}$ & to sum/ scalar multiply geom vectors, suffice do so on coords.

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**Solution.** Take $\mathbf{i}$ pointing east and $\mathbf{j}$ pointing north & units are km.

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**E.g.** Are $A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, C = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, D = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ the vertices of a parallelogram?

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## Definition

Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ (or are geom vectors). A *linear combination* of these vectors is a vector of the form

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**Q** Is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$?

**Eg** $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *span* the space $V$ of all 3-dim geom vectors

# Standard basis vectors for $\mathbb{R}^n$

In $\mathbb{R}^n$, the vector $\mathbf{e}_j$ is the $n$-tuple with 1 in the $j$th position and zeros elsewhere.

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Obviously, every vector in $\mathbb{R}^n$ can be written uniquely as a linear combination of $\mathbf{e}_1, \dots, \mathbf{e}_n$, eg

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3.$$

The vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ are called the *standard basis vectors* for $\mathbb{R}^n$.

# Length and distance in $\mathbb{R}^n$

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**Example.** a) What is $\left| \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right|$?

b) Suppose that the point $A$ has coordinates $(1, 2, 3)$ and the point $B$ has coordinates $(-1, 2, 5)$. What is the distance between $A$ and $B$?

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**N.B.** There are many other solutions! (What are they?)

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Write the line $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$ in Cartesian form.

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What about $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$?

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To obtain the cartesian form for a line $L$, we need 2 such equations. Each defines a plane $P_1, P_2$ and solving simultaneously gives the solution $P_1 \cap P_2$. This will be a line unless . . . .

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Usually, (but not always) we can write the 2 equations in the form

$$\frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}$$

for some constants $a_i, v_i$.

# Cartesian to parametric form for lines in $\mathbb{R}^3$

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# Why bother defining lines in $\mathbb{R}^n$?

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- In particular, you can't get a point or two points etc.
- If $n > 3$, we can use our geometric intuition to understand solutions to many equations provided we generalise our notions of things like lines in $\mathbb{R}^3$ to lines in higher dimensions.

# Planes in $\mathbb{R}^n$

## Definition

A *plane in* $\mathbb{R}^n$ is defined to be a set of the form

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The expression $\mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$ is a *parametric vector form* for *the plane through* $\mathbf{a}$ *parallel to the vectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$.

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**Q** What if $\mathbf{v}_1, \mathbf{v}_2$ above are parallel?

# Parametric form for plane determined by 3 points

## Question

Find a parametric vector equation for the plane through the points $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

# Meanwhile back to Han Solo

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**A** Without the Death Star, the flight trajectory would be the *ray* with parametric equation

# Example: Intersecting planes

**Q** Find the intersection of the plane $P_1$ with cartesian eqn $x + y - z = 1$ & the plane $P_2$ with parametric form

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

**A**