Chapter 1: Introduction to Vectors

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A typical problem

Question

600m north of Mos Eisley, Han flies dues east at 200ms^{-1}. Where is he t seconds later?

Pictorial Ans

Remark This question illustrates the two fundamental operations of vectors that we'll be studying in this chapter.

Geometric vectors and their arithmetic

A (geometric) *vector* $\boldsymbol{v}$ is often represented by an arrow or a directed line segment & encodes the data of magnitude $|\boldsymbol{v}|$ & direction.

Eg

Vector addition Given vectors $\boldsymbol{v}, \boldsymbol{w}$

Vector scalar multiplication Given a scalar $\lambda \in \mathbb{R}$

Reminder on real numbers

Primary and high school arithmetic uses numbers that we call *real*. For example, the numbers

$$0,\ 73,\ -2\frac{1}{5},\ \sqrt{2},\ \pi - e^2, \ldots .$$

and so on are real numbers.

We will denote the real number system by $\mathbb{R}$. We visualize $\mathbb{R}$ as an infinitely long line:

Within the set $\mathbb{R}$ we have

- Natural numbers, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
- Integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- Rational numbers, $\mathbb{Q} = \left\{ \frac{p}{q} \ :\ p, q \in \mathbb{Z},\ q \neq 0 \right\}$.
- Positive numbers, $\mathbb{R}^+ = \{x \in \mathbb{R} \ :\ x > 0\}$.
- Irrational numbers, i.e. those that aren't in $\mathbb{Q}$.

and many other subsets of numbers. Sometimes, we refer to numbers as *scalars* (as opposed to vectors).

Properties of vector arithmetic

Question

In what sense is vector addition a type of "addition" & vector multiplication a type of "multiplication"?

A They behave like adding/multiplying numbers $\because$

Commutative law $\boldsymbol{v} + \boldsymbol{w} =$

Associative law of addition $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} =$

Existence of zeros/negatives $\mathbf{0} + \boldsymbol{v} = \boldsymbol{v}, -\boldsymbol{v} := (-1)\boldsymbol{v}$ satisfies

Associative law of multiplication $\lambda(\mu \boldsymbol{v}) =$

Distributive law $\lambda(\boldsymbol{v} + \boldsymbol{w}) = \lambda \boldsymbol{v} + \lambda \boldsymbol{w}$ (Vector) $\quad (\lambda + \mu)\boldsymbol{v} =$

Why are these true?

Midpoints

Given points A, B in space, we get *displacement vector* $\overrightarrow{AB}$

Special case Given distinguished point O (for "origin") get *position vector*

Midpoints

Let $\boldsymbol{a}, \boldsymbol{b}$ be the position vectors for the points A, B. The midpoint of AB has position vector

$$\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a}) = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}).$$

Why?

Q Show that the diagonals of a parallelogram bisect each other.

The vector space $\mathbb{R}^n$

Let n be a positive integer. An *n-tuple* or *n-vector* is an ordered list of n numbers $a_1, a_2, \dots, a_n$, written as either a column vector or (less often in this course) a row vector:

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad \text{or} \quad \boldsymbol{a} = (a_1, a_2, \dots, a_n).$$

The set of all n-tuples is denoted $\mathbb{R}^n$.

We can define vector addition and scalar multiplication on $\mathbb{R}^n$ *coordinatewise*.

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} := \begin{pmatrix} a_1 + \\ a_2 + \\ \vdots \\ a_n + \end{pmatrix}, \qquad \lambda \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} := \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{pmatrix}.$$

Rem These operations satisfy the commutative, associative & distributive laws we saw for geometric vectors.

Co-ordinates on a plane

Fix a plane in space and let V be geometric vectors parallel to (lying on) it. Pick non-parallel $\boldsymbol{i}, \boldsymbol{j} \in V$ (later assume these unit length and orthogonal).

From co-ordinates to geometric vectors Given $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2$ get corresponding geometric vector wrt $B = \{\boldsymbol{i}, \boldsymbol{j}\}$:

Conversely,

Fact-Definition

Every geometric vector $\boldsymbol{a}$ in V can be written as $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$
for some unique pair of numbers $a_1, a_2 \in \mathbb{R}$.

The *co-ordinates or coordinate vector of* $\boldsymbol{a}$ (with respect to $\boldsymbol{i}, \boldsymbol{j}$) is $[\boldsymbol{a}]_B = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

Rem In fancy language, there's a 1-1 correspondence between $\mathbb{R}^2$ and V.

Co-ordinate arithmetic reflects vector arithmetic

Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j},\ \boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ so coords are $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

The coords of

$$\boldsymbol{a} \pm \boldsymbol{b} = (a_1\boldsymbol{i} + a_2\boldsymbol{j}) \pm (b_1\boldsymbol{i} + b_2\boldsymbol{j}) = (a_1 \pm b_1)\boldsymbol{i} + (a_2 \pm b_2)\boldsymbol{j}$$

are $\begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \end{pmatrix}$.

Sim, the coords of $\lambda\boldsymbol{a} = \lambda(a_1\boldsymbol{i} + a_2\boldsymbol{j}) = \lambda a_1\boldsymbol{i} + \lambda a_2\boldsymbol{j}$ are $\begin{pmatrix} \lambda a_1 \\ \lambda a_2 \end{pmatrix}$.

Upshot This says to sum, subtract or multiply vectors, we need only sum, subtract or multiply coords.

3-dim version Sim if you pick mutually orthogonal, unit length vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ in space, $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ are the coordinates of $a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_2\boldsymbol{k}$ & to sum/ scalar multiply geom vectors, suffice do so on coords.

Example

To find coords recall given a right angle triangle,

Question

I walk 1km due west, then 4km on a bearing 30° east of north. Where do I end up?

Solution. Take $\boldsymbol{i}$ pointing east and $\boldsymbol{j}$ pointing north & units are km.

Simple geometric applications of vectors

Given $O, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ coords of point P in space are coords of $\overrightarrow{OP}$. Below, we're lazy & confuse points/vectors with their coords.

E.g. Are $A = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, B = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, C = \begin{pmatrix} 6 \\ -2 \\ 5 \end{pmatrix}$ collinear?

E.g. Are $A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, C = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, D = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ the vertices of a parallelogram?

Linear combinations and span

Definition

Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ (or are geom vectors). A *linear combination* of these vectors is a vector of the form

$$\lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2 + \cdots + \lambda_k \boldsymbol{v}_k \quad \text{with} \quad \lambda_1, \ldots, \lambda_k \in \mathbb{R}\,.$$

The set of all of these is called the *span* & denoted

Q Is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$?

Eg $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ *span* the space V of all 3-dim geom vectors

Standard basis vectors for $\mathbb{R}^n$

In $\mathbb{R}^n$, the vector $\boldsymbol{e}_j$ is the n-tuple with 1 in the jth position and zeros elsewhere.

$\mathbb{R}^2$: $\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$\mathbb{R}^3$: $\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Obviously, every vector in $\mathbb{R}^n$ can be written uniquely as a linear combination of $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$, eg

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 + x_3 \boldsymbol{e}_3.$$

The vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ are called the *standard basis vectors* for $\mathbb{R}^n$.

Length and distance in $\mathbb{R}^n$

Pythagoras' thm $\Longrightarrow$ the length of a geometric vector with coords $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is $\sqrt{a_1^2 + a_2^2}$. This suggests the following generalisation of the length concept to $\mathbb{R}^n$.

Definition

Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$.

- The *length* of $\boldsymbol{a}$ is defined to be $|\boldsymbol{a}| = \sqrt{a_1^2 + \cdots + a_n^2}$.
- the *distance* between $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined to be $\operatorname{dist}(\boldsymbol{a}, \boldsymbol{b}) = |\boldsymbol{b} - \boldsymbol{a}|$.

Example. a) What is $\left| \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right|$?

b) Suppose that the point A has coordinates $(1, 2, 3)$ and the point B has coordinates $(-1, 2, 5)$. What is the distance between A and B?

Lines in $\mathbb{R}^n$

A line L in 2 or 3-dim space is determined by

- a point A on the line, say with position vector $\boldsymbol{a} = \overrightarrow{OA}$ and,
- a direction, say given by a non-zero vector $\boldsymbol{v}$

The general point of L ought to be:

$$\boldsymbol{x} = \boldsymbol{a} + \lambda \boldsymbol{v}, \ \lambda \in \mathbb{R}$$

This is called the *parametric vector form* of the line L. We call λ the parameter, and as it varies over $\mathbb{R}$, the variable $\boldsymbol{x}$ varies over all the points of the line L.

Definition

A *line* in $\mathbb{R}^n$ is any set of the form

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{a} + \lambda \boldsymbol{v}, \ \lambda \in \mathbb{R}\}$$

for some fixed vectors $\mathbf{0} \neq \boldsymbol{v}, \boldsymbol{a} \in \mathbb{R}^n$. Note $\boldsymbol{a}$ gives a point on the line and $\boldsymbol{v}$ its direction.

Finding parametric forms for lines from Cartesian form

In high school, you express a line in the plane in Cartesian form $ax + by = c$.

Question

Find a parametric vector form for the line $y = 3x + 2$ in $\mathbb{R}^2$.

A We introduce the parameter $\lambda =$

N.B. There are many other solutions! (What are they?)

Parametric to Cartesian form

Question

Write the line $\boldsymbol{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$ in Cartesian form.

The secret is to eliminate the extra variable λ!

Solution. Write $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ -1 + \lambda \end{pmatrix}$. Then

$$\lambda =$$

What about $\boldsymbol{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$?

Cartesian form for lines and planes in $\mathbb{R}^3$

Recall that a plane in xyz-space can be described by an equation

$$ax + by + cz = d$$

where not all a, b, c are 0. This is called the *Cartesian form* for the plane. The terms in this equation can of course be re-arranged many ways (see below).

To obtain the cartesian form for a line L, we need 2 such equations. Each defines a plane P_1, P_2 and solving simultaneously gives the solution $P_1 \cap P_2$. This will be a line unless . . . .

Usually, (but not always) we can write the 2 equations in the form

$$\frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}$$

for some constants a_i, v_i.

Cartesian to parametric form for lines in $\mathbb{R}^3$

Question

Find the parametric form for $\dfrac{x-2}{3} = \dfrac{y+1}{6} = \dfrac{z-3}{-2}$.

A We can find a point on the line and a direction vector or just introduce the parameter

Parametric to cartesian form for lines in $\mathbb{R}^3$

Question

Find a cartesian form for

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

What about $\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}$?

Why bother defining lines in $\mathbb{R}^n$?

- Suppose we are solving equations in n unknowns $x_1, \ldots, x_n$. If $n = 3$, it is often good to visualise the solution set in $x_1x_2x_3$-space.
- For example, solving simultaneously

$$a_1x_1 + a_2x_2 + a_3x_3 = a, b_1x_1 + b_2x_2 + b_3x_3 = b$$

  should on geometric grounds, give either a line, plane or the empty set.
- In particular, you can't get a point or two points etc.
- If $n > 3$, we can use our geometric intuition to understand solutions to many equations provided we generalise our notions of things like lines in $\mathbb{R}^3$ to lines in higher dimensions.

Planes in $\mathbb{R}^n$

Definition

A *plane in* $\mathbb{R}^n$ is defined to be a set of the form

$$S = \{\boldsymbol{a} + \lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2 | \lambda_1, \lambda_2 \in \mathbb{R}\},$$

where $\boldsymbol{a}$, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are fixed vectors in $\mathbb{R}^n$, and $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are not parallel.

The expression $\boldsymbol{x} = \boldsymbol{a} + \lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$ is a *parametric vector form* for *the plane through* $\boldsymbol{a}$ *parallel to the vectors* $\boldsymbol{v}_1$ *and* $\boldsymbol{v}_2$.

The above picture shows that when $n = 3$, our definition agrees with our old one.

Q What if $\boldsymbol{v}_1, \boldsymbol{v}_2$ above are parallel?

Parametric form for plane determined by 3 points

Question

Find a parametric vector equation for the plane through the points $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

Meanwhile back to Han Solo

Question

The Millenium Falcon, at coords $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is flying in direction $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Will it hit the Death Star wall, a plane with Cartesian eqn $x - y - z = 1$?

A Without the Death Star, the flight trajectory would be the *ray* with parametric equation

Example: Intersecting planes

Q Find the intersection of the plane P_1 with cartesian eqn $x + y - z = 1$ & the plane P_2 with parametric form

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

A