Lecture 9: Coordinates. Linear Dependence

Aim Lecture Set up theory for putting coords on any vect space. These allow you to compute in any vect space.

Informal first look at coordinates

Consider non-parallel $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$

$W =$ the plane $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

Set up "grid" or "coordinate" system

$\mathbf{v}_1, \mathbf{v}_2$ determines coords on W

$\mathbf{v}_1$ has coords

$\mathbf{v}_2$ has coords

$2\,\mathbf{v}_1 + \mathbf{v}_2$ has coords

$\lambda\,\mathbf{v}_1 + \mu\,\mathbf{v}_2$ has coords

Basic Idea: A coord system on vect space determined by "unit" vectors along positive axes.

BUT not every set of vectors in V give a coord system on V.

e.g. above example fails if $\mathbf{v}_1, \mathbf{v}_2$ parallel.

Q Which sets of vectors give legitimate coord system?

Linear dependence

Defn 1 Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq V =$ vect space / field $\mathbb{F}$.

S is linearly dependent if there are

scalars $\lambda_1, \ldots, \lambda_n$ not all 0,

with $\lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n = \mathbf{0}$.

i.e. some non-trivial linear combn of S is zero.

e.g. 1 $V = \mathbb{R}^2 \ni S = \{(1,0), (0,1), (3,2)\}$

Note $(3,2) = 3(1,0) + 2(0,1)$

$\implies 1(3,2) - 3(1,0) - 2(0,1) = (0,0)$

so S is lin depend.

e.g. 2 $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are lin depend iff they are parallel.

Why? If $\mathbf{v}, \mathbf{w}$ are lin depend so

$\lambda \mathbf{v} + \mu \mathbf{w} = \mathbf{0}$ for some λ, μ not both 0, then

either $\lambda \neq 0$ so $\mathbf{v} =$

or $\mu \neq 0$ so $\mathbf{w} =$

Either way $\mathbf{v} \parallel \mathbf{w}$.

Conversely, if $\mathbf{v} \parallel \mathbf{w}$ so say $\mathbf{v} = \lambda \mathbf{w}$

then $1\mathbf{v} - \lambda \mathbf{w} = \mathbf{0}$ so $\mathbf{v}, \mathbf{w}$ are lin depend.

Defn 2 $S \subseteq V$ = vect space / field $\mathbb{F}$. Say S is linearly independent if it is not lin dependent.

i.e. If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ then the only soln to

$\lambda_1 \mathbf{v}_1 + \ldots \lambda_n \mathbf{v}_n = \mathbf{0}$

is $0 = \lambda_1 = \ldots = \lambda_n$.

Coordinates

Thm (Uniqueness of Linear Combn)

Let V = vect space / field $\mathbb{F}$ & S

$= \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ be non-empty & lin indep.

Any vector $\mathbf{v} \in \text{Span}(S)$ can be written uniquely as $\mathbf{v} = \lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n$

i.e. If also $\mathbf{v} = \lambda_1' \mathbf{v}_1 + \ldots + \lambda_n' \mathbf{v}_n$

then $\lambda_1 = \lambda_1', \ldots, \lambda_n = \lambda_n'$.

Note: $\lambda_1, \ldots, \lambda_n$ give coordinates of $\mathbf{v}$.

Proof: Suppose

$\mathbf{v} = \lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n$

$= \lambda_1' \mathbf{v}_1 + \ldots + \lambda_n' \mathbf{v}_n$

Subtracting using distributive, associative and commutative laws gives

$\mathbf{0} = (\lambda - \lambda_1') \mathbf{v}_1 + \ldots + (\lambda_n - \lambda_n') \mathbf{v}_n$

Defn of lin indep $\implies$

Defn 3 If $B = \{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is an ordered lin indep set & $\mathbf{v} \in \text{Span}(B)$ then the coord of $\mathbf{v}$ wrt B is

$[\mathbf{v}]_B := (\lambda_1, \ldots, \lambda_n)^T$

where $\lambda_1, \ldots, \lambda_n$ are the unique scalars in

$\mathbf{v} = \lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n$

e.g. 3 $B = \{1, x, x^2\}$ is lin indep

$\because \lambda_1 1 + \lambda_2 x + \lambda_3 x^2 = 0 \implies 0 = \lambda_1 = \lambda_2 = \lambda_3.$

$[a_0 + a_1 x + a_2 x^2]_B = (a_0, a_1, a_2)^T.$

N.B. Lin indep is essential in defn of coords.

Testing lin depend in $\mathbb{F}^m$

As in lecture 9, consider vectors in $\mathbb{F}^m$

$$\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \ldots, \mathbf{a}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$A = (\mathbf{a}_1\, \mathbf{a}_2 \ldots \mathbf{a}_n)$.

Saw $x_1\, \mathbf{a}_1 + \ldots + x_n\, \mathbf{a}_n = A\, \mathbf{x}$ hence,

Propn $B = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is lin independent iff $A\, \mathbf{x} = \mathbf{0}$ has unique soln $\mathbf{x} = \mathbf{0}$. If $\mathbf{v} \in col(A)$ then $[\mathbf{v}]_B$ is soln $\mathbf{x}$ to $A\, \mathbf{x} = \mathbf{v}$.

E.g. 4 $\mathbf{v}_1 = (1, 1, 2)^T, \mathbf{v}_2 = (2, 1, 1)^T, \mathbf{v}_3 = (1, 2, 5)^T, \mathbf{v}_4 = (0, 0, 1)^T$

Is $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ lin independent?

i.e. Is $\mathbf{x} = \mathbf{0}$ the unique soln to

$x_1\, \mathbf{v}_1 + x_2\, \mathbf{v}_2 + x_3\, \mathbf{v}_3 + x_4\, \mathbf{v}_4 = \mathbf{0}$?

Re-writing with matrices, we solve $A\,\mathbf{x} = \mathbf{0}$ where $A = (\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4)$.

Third column not leading so there are infinitely many solns (1 parameters worth).

Soln not unique $\implies$ S not lin indep.

In fact $\mathbf{x} = (-3, 1, 1, 0)^T$ is a non-zero soln to $A\,\mathbf{x} = \mathbf{0}$ which corresponds to

the non-trivial relation

Also, above calculation shows $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ lin indep

Why? Omitting 3rd column from above calculation cont'd

all columns leading so the unique soln to

$x_1\,\mathbf{v}_1 + x_2\,\mathbf{v}_2 + x_4\,\mathbf{v}_4 = \mathbf{0}$

is $x_1 = x_2 = x_4 = 0$ i.e. $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$ lin indep.

Finding coordinates

e.g. 4 cont'd $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$. Let $\mathbf{v} = (1, 2, 1)$. Is $\mathbf{v} \in \mathrm{Span}(B)$ and if so what is $[\mathbf{v}]_B$?

A Need to solve $\lambda_1\,\mathbf{v}_1 + \lambda_2\,\mathbf{v}_2 + \lambda_3\,\mathbf{v}_4 = \mathbf{v}$.

$[\mathbf{v}]_B$

Working in other vector spaces

Reduce question to linear algebra question of the type in e.g. 2 as in the following

E.g. 5 Is $B = \{p_1(x) = 1 + x + 2x^2, p_2(x) = 2 + x + x^2, p_3(x) = x^2\}$ lin independent in $\mathbb{P}$? If so what is coord of $1 + 2x + x^2$ wrt B?

A Try to solve

(*) $\lambda_1(1 + x + 2x^2) + \lambda_2(2 + x + x^2) + \lambda_3 x^2 = 1 + 2x + x^2$

Equate coefficients:

const:

x:

x^2:

This gives the same system of linear equations as in E.g. 4.