Lecture 7: Subspaces. Linear combinations

Aim Lecture Intro subspaces which allow you to define linear subsets of a vect space.

Subspace axioms

Subspace **Thm-Defn** Let V be a vector space / field $\mathbb{F}$. A subset $W \subseteq V$ is a subspace of V if (all) the following closure axioms hold.

a. W contains the zero vector.

b. W is closed under addn

i.e. for any $\mathbf{v}, \mathbf{w} \in W$ we have $\mathbf{v} + \mathbf{w} \in W$.

& c. W is closed under scalar multn

i.e. for any $\mathbf{w} \in W, \lambda \in \mathbb{F}$ we have $\lambda \mathbf{w} \in W$.

In this case, addn & scalar multn on V restrict to addn and scalar multn law on W making W a vector space.

We write $W \leq V$ in this case.

Why? Vector space axioms for $V \implies$ those for W. Try to check some axioms for yourself.

Counter-examples

e.g. 1 Subspaces aren't curved.

$V = \mathbb{R}^2, \quad W = \{(x, y)|x^2 + y^2 = 2x + 2y\}$.

W is not a subspace of V

Why? Note $(2, 0) \in W$,

but $2(2, 0) = (4, 0) \notin W$.

$\therefore W$ is not closed under

scalar multn.

W fails one of the axioms of a subspace.

$\therefore W$ is not a subspace.

Note: If $W \leq \mathbb{R}^n$ contains $\mathbf{w} \neq \mathbf{0}$, it contains the line $\mathbf{x} = \lambda\,\mathbf{w}, \lambda \in \mathbb{R}$.

e.g. 2 $V = \mathbb{R}^2, \quad W = x$ and y axes is not a subspace.

Why? $(1,0),(0,1) \in W$

but $(1,0)+(0,1)=(1,1) \notin W$

$\therefore W$ is not closed under addn.

$\therefore W$ is not a subspace.

Note W is closed under scalar multn.

ex Is $\mathbb{Z}$ a subspace of $\mathbb{R}$?

ex Is $\{(x,y)|x+2y=3\}$ a subspace of $\mathbb{R}^2$?

Examples

e.g. 3 Suppose $\mathbf{v} \in \mathbb{R}^3$ represents some portfolio i.e. $\mathbf{v}=(v_1,v_2,v_3)$ where

$v_i =$ amount of investment in i-th asset.

Suppose $r_i =$ rate of return of i-th asset.

so net profit is $r_1 v_1 +$

Show $W := \{\mathbf{v} \in \mathbb{R}^3 \,|\text{net profit is } 0\}$

is a subspace of $\mathbb{R}^3$.

A $W = \{\mathbf{v} \in \mathbb{R}^3 \,|\mathbf{r}.\,\mathbf{v} = 0\}$

where $\mathbf{r} = (r_1, r_2, r_3)$.

N.B. Geom, W is a

Check closure axioms.

Zero: $\mathbf{r}\,.\mathbf{0} = 0$ so $\mathbf{0} \in W$.

Addn: if $\mathbf{v}, \mathbf{w} \in W$ then

$\mathbf{r}\,.(\mathbf{v} + \mathbf{w}) = \mathbf{r}\,.\,\mathbf{v} + \mathbf{r}\,.\,\mathbf{w} = 0 + 0 = 0$

so $\mathbf{v} + \mathbf{w} \in W$ & W is closed under addn.

Scalar Multn: if $\mathbf{v} \in W, \lambda \in \mathbb{R}$ then

$\mathbf{r}\,.(\lambda\,\mathbf{v}) = \lambda\,\mathbf{r}\,.\,\mathbf{v} = \lambda 0 = 0$

so $\lambda\,\mathbf{v} \in W$ & W is closed under scalar multn.

Closure axioms hold so $W \leq \mathbb{R}^3$.

e.g. 4 $V = \mathcal{R}[\mathbb{R}]$ has subsets

$\mathbb{P}$ = set of real poly fns.

$\mathbb{P}_d$ = subset of poly of degree $\leq d$.

$\mathbb{P}, \mathbb{P}_d$ are subspaces of $\mathcal{R}[\mathbb{R}]$.

Why? 0 is a poly of degree $\leq d$ so $0 \in \mathbb{P}_d$.

$\mathbb{P}_d$ is closed under addn $\because$

$\mathbb{P}_d$ is closed under scalar multn since

This verifies closure axioms for a subspace so $\mathbb{P}_d \leq \mathcal{R}[\mathbb{R}]$

Argument above works for $d = \infty$ to show also $\mathbb{P} \leq \mathcal{R}[\mathbb{R}]$.

e.g. 5 Sim $\mathbb{P}(\mathbb{C})$ resp $\mathbb{P}_d(\mathbb{C})$ = set of complex poly (resp of degree $\leq d$) are subspaces of $\mathcal{C}(\mathbb{C})$.

Alternative Subspace Thm Let V = vect space / field $\mathbb{F}$. Then $W \subseteq V$ is a subspace iff i)

$\mathbf{0} \in W$ and ii) for any $\mathbf{v}, \mathbf{w} \in W, \lambda \in \mathbb{F}$ we have $\lambda\,\mathbf{v} + \mathbf{w} \in W$.

Why? Condn ii) certainly follows from closure under addn & scalar multn. Conversely, condn ii) specialises to these two closure axioms on setting $\lambda = 1$ or $\mathbf{w} = \mathbf{0}$ (which is allowed by condn i)).

e.g. 6 Interval $I = [a, b] \subseteq \mathbb{R}, V = \mathcal{R}[I]$.

Let $C(I)$ = set of continuous fns.

$C^{(k)}(I)$ = set of k-times differentiable fns s.t. $f^{(k)}(x)$ is also continuous.

Then $C(I), C^{(k)}(I)$ are subspaces.

e.g. We check $C(I)$ is a subspace using the alternative subspace thm.

i) 0 fn is continuous so $0 \in C(I)$.

ii) Let $\lambda \in \mathbb{R}, f, g \in C(I)$. Then $\lambda f + g$ is cont so $\lambda f + g \in C(I)$.

The alternative subspace thm then shows $C(I)$ is a subspace of $\mathcal{R}[I]$.

Above subspaces are useful in calculus.

Facts about subspaces

Fact Let V be vector space / field $\mathbb{F}$.

1) $\mathbf{0}$ and V are subspaces of V.

2) If $W \leq V, U \leq W$ then $U \leq W$.

3) The zero of V is the zero of any subspace.

Proof: easy. Clear from any example

$0 < \mathbb{P}_d < \mathbb{P} < C(\mathbb{R})$.

Linear Combinations

Defn Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq V$ = vect space / field $\mathbb{F}$. A linear combination of S is a vector or expression of form

$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \ldots + \lambda_n \mathbf{v}_n$

for some scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$.

If $S = \varnothing$, then we define by default that $\mathbf{0}$ is the only linear combn of S.

e.g. 7 $\mathbf{v}_1 = (2,1), \mathbf{v}_2 = (0,1)$

Write $(4,3)$ as a lin

combn of $\{\mathbf{v}_1, \mathbf{v}_2\}$.

$(4,3) = 2(2,1) + 1(0,1)$

$= 2(2,1) + (0,1)$.

Hence $(4,3)$ is a lin combn of $\mathbf{v}_1, \mathbf{v}_2$.

e.g. 8 $V = M_{22}(\mathbb{R})$

$$\mathbf{v}_1 = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}$$

$\mathbf{w}$ is not a lin combn of $\mathbf{v}_1, \mathbf{v}_2$.

Why? For $\lambda_1, \lambda_2 \in \mathbb{R}$,

$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 =$

e.g. 9 Is e^{3x} a lin combn of e^x, e^{2x}?

A No. For suppose to the contrary that

$e^{3x} = \lambda e^x + \mu e^{2x}$ for some $\lambda, \mu \in \mathbb{R}$.

$\therefore 1 =$

Take limit

RHS

Thus e^{3x} is not a lin combn of e^x, e^x.

Prop If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq W \leq V =$ vector space / field $\mathbb{F}$, then every lin combn of elements in S is in W.

Proof: For $\lambda_1, \ldots, \lambda_r \in \mathbb{F}$ we have

$\lambda_1 \mathbf{v}_1 + \ldots + \lambda_r \mathbf{v}_r$