Lecture 6: Vector Spaces

Aim Lecture Introduce vector spaces which provide natural context/language for describing linear phenomena.

Motivation: some linear phenomena

In 3D-space

Line has

param form

$\mathbf{x} = \mathbf{a} + \lambda\,\mathbf{v}, \lambda \in \mathbb{R}$.

In $\mathbb{R}^3$**:** 2 inequivalent non-zero linear equations has soln set of form

$\mathbf{x} = \mathbf{a} + \lambda\,\mathbf{v}, \lambda \in \mathbb{R}$ for some choice of $\mathbf{a}, \mathbf{v} \in \mathbb{R}^3$.

Function space Soln to $\frac{dy}{dx} = 2x$

is "line"

of quadratic

fns $y = x^2 + \lambda 1, \lambda \in \mathbb{R}$.

Wish to define notion of vector space where the expression $\mathbf{a} + \lambda\,\mathbf{v}$ makes sense so need notion of vector addn & scalar multn.

More gen, want standard rules of vector arithmetic to hold.

Vector space axioms

Let $\mathbb{F}$ be a field like $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. A vector space over $\mathbb{F}$ is

a. A set V of elements called

& b. An addition law denoted + which assigns to any $\mathbf{v}, \mathbf{w} \in V$ another vector $\mathbf{v} + \mathbf{w} \in V$,

This new vector is called

& c. A scalar multn law which assigns to any $\mathbf{v} \in V$ and $\lambda \in \mathbb{F}$ a vector $\lambda\,\mathbf{v} \in V$, such that the following standard laws of vector arithmetic hold.

For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

1. Associative Law of Addition:

$(\mathbf{u} + \mathbf{v}) + \mathbf{w} =$

2. Commutative Law of Addition:

$\mathbf{u} + \mathbf{v} =$

3. Existence of Zero: there's a vector $\mathbf{0}$ called zero which satisfies the following special (defining) property: $\mathbf{0} + \mathbf{v} = \mathbf{v}$ (for all $\mathbf{v} \in V$).

4. Existence of Negatives: there's a vector $-\mathbf{v}$ called the negative of $\mathbf{v}$ which satisfies the defining property $\mathbf{v} + (-\mathbf{v}) =$

5. Associative Law for Scalar Multn:

$(\lambda\mu)\,\mathbf{v} = \lambda(\mu\,\mathbf{v})$.

6. $1\mathbf{v} = \mathbf{v}$.

7. Scalar Distributive:

$(\lambda + \mu)\,\mathbf{v} = \lambda\,\mathbf{v} + \mu\,\mathbf{v}$

8. Vector Distributive:

$\lambda(\mathbf{v} + \mathbf{w}) =$

Subtle Pt: Zero & negatives are uniquely defined by their defining property.

See notes §7.2, propn 1.

Rem The above laws are called the axioms for a vector space.

Examples

e.g. 1 $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ when equipped with coordinate-wise addn & scalar multn i.e.

addition rule:

scalar multiplication rule:

e.g. 2 $\mathbb{C}^n = \{(z_1, \ldots, z_n) | z_1, \ldots, z_n \in \mathbb{C}\}$ is a vector space $/\mathbb{C}$ when equipped with coordinate-wise addn & scalar multn.

e.g. 3 Let V = set of geom vectors

(i.e. "arrows") in 3D-space

V is a vector space if we define

addn = usual head to tail addn of arrows

scalar multn = usual scaling "length" of vector.

e.g. 4 Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ or any other field.

Let $M_{mn}(\mathbb{F})$ be set of $m \times n$-matrices over $\mathbb{F}$.

Define

vector addn to be matrix addn

scalar multn by scalar multn of matrices

i.e.

Then $M_{mn}(\mathbb{F})$ is a vector space $/\, \mathbb{R}$.

You can check all axioms. Here we'll only check existence of zero axiom:

Hence the zero matrix satisfies the defining property of a zero in a vector space i.e. it is the zero in $M_{mn}(\mathbb{F})$ so there exists a zero vector in $M_{mn}(\mathbb{F})$. This checks the axiom.

N.B. Also, negative of a matrix is the vect space negative.

Abstract vect space defns coincide with matrix defns.

e.g. 5 $X =$ non-empty set.

$\mathcal{R}[X] :=$ set of real-valued fns on X

$\mathcal{R}[X]$ is a vector space / $\mathbb{R}$ if define vector operations pointwise i.e.

vector addn: for $f, g \in \mathcal{R}[X]$

$(f+g)(x) = f(x) + g(x)$

scalar multn: for $\lambda \in \mathbb{R}, f \in \mathcal{R}[X]$

$(\lambda f)(x) =$

Then $\mathcal{R}[X]$ is a vector space / $\mathbb{R}$ with zero the constant fn with value 0.

N.B. Abstract vect space defns coincide with calculus defns.

Sim **e.g. 6** $X =$ non-empty set.

$\mathcal{C}[X] :=$ set of $\mathbb{C}$-valued fns on X.

$\mathcal{C}[X]$ is a vector space / $\mathbb{C}$ if define

addn & scalar multn pointwise.

Properties of vector spaces

Let $V =$ vector space / field $\mathbb{F}$.

Subtraction Let $\mathbf{v}, \mathbf{w} \in V$. $-\mathbf{v}$ is the only vector s.t.

We can define $\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v})$

Vect space axioms $\Longrightarrow$ can algebraically manipulate vectors as you would geom vect.

e.g. 8 For $\mathbf{v}, \mathbf{w} \in V$ simplify

$2(3\,\mathbf{v} + 4\,\mathbf{w}) - 3\,\mathbf{w}$

$= (2(3\,\mathbf{v}) + 2(4\,\mathbf{w})) + (-3)\,\mathbf{w}$ vect distrib law

$= (6\,\mathbf{v} + 8\,\mathbf{w}) + (-3)\,\mathbf{w}$

$= 6\,\mathbf{v} + (8\,\mathbf{w} + (-3)\,\mathbf{w})$

$= 6\,\mathbf{v} + 5\,\mathbf{w}.$

ex Simplify $2(\mathbf{v} + \mathbf{w}) + 3(\mathbf{v} - \mathbf{w})$

Propn Let V be a vector space / $\mathbb{F}$. For $\lambda \in \mathbb{F}, \mathbf{v} \in V$

1. $\lambda\mathbf{0} = \mathbf{0}$ 2. $0\,\mathbf{v} =$ 3. $(-1)\,\mathbf{v} =$

4. $\lambda\,\mathbf{v} = \mathbf{0} \Longrightarrow$

Proof: 2) $0\,\mathbf{v} + 0\,\mathbf{v} = (0 + 0)\,\mathbf{v} = 0\,\mathbf{v}$

Subtract $0\,\mathbf{v}$ from both sides to see

$0\,\mathbf{v} + 0\,\mathbf{v} - 0\,\mathbf{v} = 0\,\mathbf{v} - 0\,\mathbf{v} = \mathbf{0}$

So $0\,\mathbf{v} = \mathbf{0}$.

1) is sim.

3) We check $(-1)\,\mathbf{v}$ satisfies defining property of negative i.e. $(-1)\,\mathbf{v} + \mathbf{v} = \mathbf{0}$.

$(-1)\,\mathbf{v} + \mathbf{v} =$

Hence $(-1)\,\mathbf{v} = -\,\mathbf{v}$.

4) If $\lambda \neq 0$ then

$\mathbf{v} = 1\,\mathbf{v} = (\lambda^{-1}\lambda)\,\mathbf{v} =$

Hence either $\lambda = 0$ or $\mathbf{v} = \mathbf{0}$.

An exotic example

e.g. Twisted $\mathbb{C}^n$. $V = \mathbb{C}^n$ as a set.

Addn: usual coordinate-wise addn

New twisted scalar multn: for $\lambda, z_1, \ldots, z_n \in \mathbb{C}$ define

$\lambda(z_1, \ldots, z_n) = (\bar{\lambda} z_1, \ldots, \bar{\lambda} z_n)$.

V is a vector space / $\mathbb{C}$.

Why? Axioms involving only addn checked as for usual $\mathbb{C}^n$. Let's check scalar distrib law: