Lecture 5: Stability of Dynamical Systems

Aim Lecture Introduce dynamical systems & see applicns of complex numbers to them.

Example: interest on investment

e.g. 1 An investment earns interest.

Suppose interest rate = 5%.

& interest compounded monthly

If x_n = amount of

then evolution of investment governed by

Recurrence reln $x_{n+1} = 1.05x_n, \quad n \in \mathbb{N}$.

gives soln $x_n =$

in terms of initial investment x_0.

This is an example of a discrete time system.

Continuous approximations

If you compound interest daily or per second, it's better to approximate with continuous fn.

Derive Setup Suppose $x(t) =$

& interest rate $= r$ per annum

which is compounded every Δt years.

Then $x(t + \Delta t) = x(t) + r(\Delta t)x(t)$

$rx(t) = \frac{x(t+\Delta t)-x(t)}{\Delta t}$

$\approx$

We thus see continuous compounding governed by

Differential Eqn $\frac{dx}{dt} = rx.$

It has soln $x(t) =$

This gives example of a continuous time system.

Discrete time systems

Consider sequence $x_0, x_1, x_2, \ldots$ of numbers which "evolve" according to

Difference Eqn

(*) $x_n + a_1 x_{n-1} + \ldots + a_r x_{n-r} = 0$

for $a_1, \ldots, a_r \in \mathbb{C}$.

Rem 1. (*) defines x_n in terms of

2. We say (*) defines a discrete time system.

3. More complicated dts exist.

e.g. $x_n = n$ is a soln to the diff eqn

$x_n - 2x_{n-1} + x_{n-2} = 0.$

CHECK: $x_n - 2x_{n-1} + x_{n-2} =$

Can you find another soln? Try starting with

$x_0 = x_1 = 3.$

Then $x_2 = 2x_1 - x_0 =$

Hence soln is $x_n =$

Defn The characteristic eqn of dts (*) is

$\lambda^r + a_1\lambda^{r-1} + \ldots + a_r = 0.$

Prop Let $\alpha \in \mathbb{C}$ be a root of the char eqn of (*) with multiplicity m.

Then for $k = 0, \ldots, m-1, A \in \mathbb{C}$ we have

$x_n = An^k\alpha^n$

is a soln to the dts (*).

Proof: (case $d = 2$ only). If $k = 0$ then

$x_n + a_1x_{n-1} + a_2x_{n-2} =$

$= A\alpha^{n-2}($

since α is a root of the char eqn $\lambda^2 + a_1\lambda + a_2 = 0.$

$\therefore x_n = A\alpha^n$ is a soln to the dts(*).

If $k = 1$ then char eqn is

$(\lambda - \alpha)^2 =$

$x_n + a_1x_{n-1} + a_2x_{n-2}$

$= x_n - 2\alpha x_{n-1} + \alpha^2 x_{n-2}$

$=$

$= A\alpha^n($

$\therefore x_n = An\alpha^n$ is a soln to dts (*) in this case.

Stability

Defn A soln $\{x_n\}$ to the dts (*) is

stable if $|x_n| \longrightarrow 0$ as $n \longrightarrow \infty$.

unstable if $|x_n| \longrightarrow$

We don't define stability for other asymptotic behaviour.

e.g. 2 If $x_n = n^k \alpha^n$ for some $k \geq 1, \alpha \in \mathbb{C}$

then $|x_n| = n^k |\alpha|^n$.

If $|\alpha| < 1$ then $|x_n| \longrightarrow 0$ as $n \longrightarrow \infty$ so $\{x_n\}$ is stable.

If $|\alpha| > 1$

Fact The solns to the dts(*) are precisely the sums of the solns given in the propn.

No Proof.

e.g. 3 Are the solns to the dts

$2x_n - x_{n-1} + x_{n-2} = 0$ stable?

A Char eqn is $2\lambda^2 - \lambda + 1 = 0$.

Roots are $\lambda = \frac{1 \pm \sqrt{1-8}}{4} =$

If $\alpha_\pm$ denote the roots then fact $\Longrightarrow$ any soln has form

$x_n = A(\frac{1+i\sqrt{7}}{4})^n + B(\frac{1-i\sqrt{7}}{4})^n$

for some choice of constants $A, B \in \mathbb{C}$.

$|\alpha_\pm| = \frac{1}{4}\sqrt{1+7} = \frac{\sqrt{8}}{4} < 1$

so $|x_n| \longrightarrow 0$ as $n \longrightarrow 0$ & all solns are stable.

Upshot: a) All solns to dts are stable if every root α of the char eqn has $|\alpha| < 1$.

b) If there's some root α with $|\alpha| > 1$ then there are unstable solns.

e.g. 4 $x_n + x_{n-1} + x_{n-3} + 2x_{n-4} = 0$

has unstable solns.

Why? Char eqn is

so product of roots is

& there's a root α with $|\alpha| > 1$.

Thus $x_n = \alpha^n$ is an unstable soln.

Continous time systems

Consider fns $x(t)$ which "evolve" according to

Differential eqn

(†) $\frac{d^r x}{dt^r} + a_1 \frac{d^{r-1} x}{dt^{r-1}} + \ldots + a_r x = 0.$

Rem 1. (†) defines an r-th order differential eqn.

2. It's char eqn is

$\lambda^r + a_1 \lambda^{r-1} + \ldots + a_r = 0.$

In calculus you'll see like in dts case we have,

Thm The soln to the cts (†) are sums of fns of form

$$x(t) = At^k e^{\alpha t}$$

where α is a root of the char eqn of multiplicity $> k$ and $A \in \mathbb{C}$ is some constant.

Rem again we'll say the soln $x(t)$ is

stable if $|x(t)| \longrightarrow 0$ as $t \longrightarrow \infty$.

& unstable if

e.g. 5 Let $\alpha = a + bi, a, b \in \mathbb{R}, x(t) = t^k e^{\alpha t}$.

$|t^k e^{\alpha t}| = |t|^k |e^{at+bit}| = |t|^k |e^{at}|$.

If Re $\alpha = a > 0$ then $|x(t)| \longrightarrow \infty$ as $t \longrightarrow \infty$.

If Re $\alpha = a < 0$ then

Upshot: a) All solns to cts are stable if every root α of the char eqn has Re $\alpha < 0$.

b) If there's some root α with Re $\alpha > 0$ then there are unstable solns.

e.g. 6 $\frac{d^4x}{dt^4} - 2\frac{d^3x}{dt^3} + x = 0$

has unstable solns.

Why? Char eqn is

$\implies$ there's a root α with