Lecture 4: Complex Polynomials

Aim Lecture Relate roots, factors & coefficients of poly. Factorise real & complex poly.

Remainder & Factor thm

Defn A complex polynomial of degree n is a fn $p : \mathbb{C} \longrightarrow \mathbb{C}$ of the form

$$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

for some $a_0, \ldots, a_n \in \mathbb{C}$ with $a_n \neq 0$.

If the coeff $a_0, \ldots, a_n$ are real then we say $p(z)$ is a real polynomial.

Real poly will also refer to the real-valued fn $p : \mathbb{R} \longrightarrow \mathbb{R}$ obtained by restricting the domain to $\mathbb{R}$.

Remainder Thm Let $p(z)$ be a poly & $\alpha \in \mathbb{C}$. The remainder r on dividing $p(z)$ by $z - \alpha$ is $r = p(\alpha)$.

Proof If quotient is $q(z)$ then

$p(z) =$

$\therefore p(\alpha) =$

An immediate corollary is

Factor Thm Let $p(z)$ be a poly & $\alpha \in \mathbb{C}$

Then $z - \alpha$ is a factor of $p(z)$ iff $p(\alpha) = 0$

i.e. α is a root of $p(z)$.

Factorising over $\mathbb{C}$

Fund Thm of Algebra (Gauss) Let $p(z) = a_n z^n + \ldots + a_1 z + a_0$ be a complex poly of degree $n > 0$. Then $p(z)$ has a complex root, so applying factor thm and induction we see we can express

$$(*) \quad p(z) = a_n(z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_n)$$

where $\alpha_1, \ldots, \alpha_n$ are all the roots of $p(z)$ (somtimes repeated).

Furthermore, the factorisation in (*) is unique up to permuting factors.

Defn The number of times a root α_i occurs in the factorisation is called the multiplicity of the root.

e.g. 1 $z^4 + 2z^2 + 1$

So $i, -i$ are roots of multiplicity 2.

e.g. 2 Factorise $p(z) = 4 - z^6$ over $\mathbb{C}$.

A Use thm. Find solns to $p(z) = 0$ or equivalently $z^6 = 4$.

$|z|^6 =$

$6 \operatorname{Arg} z =$

$\implies \operatorname{Arg} z =$

So roots are $z = \sqrt[3]{2}e^{\pm 2\pi i/3}$, $\sqrt[3]{2}e^{\pm \pi i/3}$,

$\sqrt[3]{2}$, $\sqrt[3]{2}e^{i\pi} = -\sqrt[3]{2}$.

$$4 - z^6 = -(z - \sqrt[3]{2}e^{2\pi i/3})(z - \sqrt[3]{2}e^{-2\pi i/3})$$
$$\times(z - \sqrt[3]{2}e^{\pi i/3})(z - \sqrt[3]{2}e^{-\pi i/3})(z - \sqrt[3]{2})(z + \sqrt[3]{2})$$

Factorising over $\mathbb{R}$

Use

Propn a) Let $p(z) = \sum a_j z^j$ be a real poly & $z = \alpha$ be a complex root. Then $\bar{\alpha}$ is also a root.

b) $(z - \alpha)(z - \bar{\alpha}) = z^2 - (2\text{Re } \alpha)z + |\alpha|^2$

which is real.

Proof 1): If $0 = p(\alpha) =$

then $0 = \sum$

e.g. 2 cont'd Factorise $4 - z^6$ over $\mathbb{R}$.

A Collect factors corresp to complex conjugate roots.

$(z - \sqrt[3]{2}e^{\pi i/3})(z - \sqrt[3]{2}e^{-\pi i/3}) =$

$z^2 - (2\text{Re } \sqrt[3]{2}e^{\pi i/3})z + |\sqrt[3]{2}e^{\pi i/3}|^2$

$= z^2 - \sqrt[3]{2}z + \sqrt[3]{4}.$

Sim $(z - \sqrt[3]{2}e^{2\pi i/3})(z - \sqrt[3]{2}e^{-2\pi i/3}) =$

$4 - z^6 = -(z - \sqrt[3]{2})(z + \sqrt[3]{2})$

$\times(z^2 - \sqrt[3]{2}z + \sqrt[3]{4})(z^2 + \sqrt[3]{2}z + \sqrt[3]{4}).$

Rem This procedure shows you can factorise any real poly into real linear & quadratic factors.

Application to polynomial interpolation

Corollary a) If poly $p(z), q(z)$ have degrees $\leq n$ & agree on $n+1$ different values $z = \alpha_1, \dots, \alpha_{n+1}$

i.e. $p(\alpha_1) =$

then $p(z) = q(z)$.

b) Any 2 poly which agree on an infinite set are the same.

Proof: Clear a) $\implies$ b). Note $g(z) := p(z) - q(z)$

has degree $\leq n$.

It also has more than n roots (namely, $\alpha_1, \ldots, \alpha_{n+1}$) so it must be 0.

e.g. 2 Given 3 distinct points $(x_1, y_1), (x_2, y_2)$ & (x_3, y_3), there is at most 1 parabola of the form $y = p(x)$ going through those points.

Why? If $y = q(x)$ also went through those points then

Symmetric polynomials in the roots

Defn A poly $p(x_1, \ldots, x_n)$ in var $x_1, \ldots, x_n$ is symmetric if it remains the same on swapping any 2 variables.

e.g. 4 In 3 var,

$x_1 + x_2 + x_3$ is symmetric

$x_1x_2 + x_2x_3$ is not because

e.g. 5 Suppose $z^2 + bz + c$ has roots $\alpha, \beta \in \mathbb{C}$.

$z^2+bz+c = (z-\alpha)(z-\beta) = z^2-(\alpha+\beta)z+\alpha\beta$.

$\implies$ sum of roots =

& product of roots =

More generally,

Prop Let $\alpha_1, \ldots, \alpha_n$ be the roots (with multiplicity) of

$p(z) = a_0 + a_1 z + \ldots + a_n z_n$.

Then $\frac{a_{n-j}}{a_n} = (-1)^j$ sum of all products of j roots.

Proof: Just expand

N.B. The a_i's are symmetric poly in the α_i's since $p(z)$ remains the same on swapping any two linear factors in its factorisation.

Thm Any symmetric poly in the roots of $p(z)$ is a poly in the coeff of $p(z)$.

No proof.

e.g. 6 If $z^3 + 2z^2 + 3z + 4$ has roots $\alpha_1, \alpha_2, \alpha_3$ then

$\alpha_1^2 + \alpha_2^2 + \alpha_3^2$ is symmetric and equals