Lecture 3: Trig identities. Complex loci.

Aim Lecture See how complex numbers can demystify some trig identities. Plot some complex loci.

Basic Formulae

Facts 1.$\overline{e^{i\theta}} = e^{-i\theta}$

2. $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

3. $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

Proof 1. From picture

2. (3. is similar)

$e^{i\theta} - e^{-i\theta} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) = 2i\sin\theta$

$\therefore \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.

Trigonometric polynomials

Defn A trigonometric polynomial is a fn of the form

$T(\theta) = a_0 + \sum_{n=1}^{N}(a_n \cos(n\theta) + b_n \sin(n\theta))$

for some $a_n, b_n \in \mathbb{R}$.

Rem: 1. If $N = \infty$, we call it a trigonometric series.

2. $T(\theta)$ is periodic & 2π is a period.

You can convert any poly in $\cos\theta$ & $\sin\theta$ into a trig poly.

e.g. 1 Write $\sin^4\theta$ as a trig poly.

A Use fact 2 and

Binomial Thm

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

e.g. $\binom{4}{0} =$, $\binom{4}{1} =$

$\sin^4\theta = \left[\frac{1}{2i}(e^{i\theta} - e^{-i\theta})\right]^4$

$= \frac{1}{16i^4}(\binom{4}{0}e^{i4\theta} - \binom{4}{1}e^{i3\theta}e^{-i\theta}+$

$= \frac{1}{16}(e^{i4\theta} + e^{-i4\theta}) - \frac{4}{16}(e^{i2\theta} + e^{-i2\theta}) + 6$

$= \frac{1}{8}\cos 4\theta -$

N.B. $\sin^4\theta$ is an even

Uses $\int \sin^4\theta d\,\theta$

Rem Fourier theory (taught in 2nd yr) shows that any nice fn of period 2π can be expanded using

e.g. 2 Write $\sin 3\theta, \cos 3\theta$ as a poly of $\sin\theta, \cos\theta$.

A De Moivre $\implies$

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3(\cos^2\theta)i\sin\theta + 3(\cos\theta)i^2\sin^2\theta + i^3\sin^3\theta$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

Equate real & imag parts

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\sin 3\theta =$

The answer is not

$\cos 3\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= 4\cos^3\theta - 3\cos\theta.$

Harder example

e.g. 3 Find

$\Sigma := \cos\theta + \cos 3\theta + \ldots + \cos(2n+1)\theta.$

A $\Sigma = \mathrm{Re}\,\left(e^{i\theta} + e^{i3\theta} + \ldots + e^{i(2n+1)\theta}\right)$

But the sum of the GP with common ratio $e^{2i\theta}$

$\Sigma' := e^{i\theta} + e^{i3\theta} + \ldots + e^{i(2n+1)\theta}$

$= \frac{e^{i(2n+3)\theta} - e^{i\theta}}{e^{2i\theta} - 1}$

Now $e^{i2\theta} - 1 = e^{i\theta}(e^{i\theta} - e^{-i\theta})$

$= e^{i\theta}2i\sin\theta.$

$e^{i(2n+3)\theta} - e^{i\theta} = e^{i(n+2)\theta}\left(e^{i(n+1)\theta} - e^{-i(n+1)\theta}\right)$

Dividing these two gives

$\Sigma' = e^{i(n+1)\theta} \frac{\sin(n+1)\theta}{\sin\theta}$

$\Sigma = \text{Re } \Sigma' = \cos(n+1)\theta\frac{\sin(n+1)\theta}{\sin\theta}$

Complex loci

If we represent $z = a + bi, w = c + di$ as the vectors $(a, b), (c, d)$ then $z - w = (a - c) + i(b - d)$ is represented by the vector $\mathbf{v}$ from w to z in Argand diagram.

Note: For vector $\mathbf{v}$ as in picture

1. $|\,\mathbf{v}\,| = |z - w|$
2. Dirn of $\mathbf{v}$ is given by $\text{Arg}\, z - w$.

see MATLAB geom2.m file

Triangle Inequality For $u, v, w \in \mathbb{C}$

$|u - w| \leq |u - v| + |v - w|$

e.g. 4 Sketch $S := \{z \in \mathbb{C} \,|\, \text{Im } z = |z - i|\}$

i.e. locus of points equidistant from the real axis and i.

A Consider $z = x + iy \in S, x, y, \in \mathbb{R}$.

Im $z = y$.

$|z - i| = |x + (y - 1)i| = \sqrt{x^2 + (y - 1)^2}$

Equating and squaring gives

$y^2 = x^2 + y^2 - 2y + 1$.

Thus $y = \frac{1}{2}(x^2 + 1)$.

e.g. 5 Sketch $0 \leq \text{Arg } z^3 \leq \frac{\pi}{2}$

A $\text{Arg}\, z^3 = 3\,\text{Arg}\, z + 2n\pi$ for some $n \in \mathbb{Z}$.

$\therefore 0 \leq 3\text{Arg } z \leq \frac{\pi}{2} \Longrightarrow$

OR $-2\pi \leq 3\text{Arg } z \leq \frac{-3\pi}{2} \Longrightarrow$

OR $2\pi \leq 3\text{Arg } z \leq \frac{5\pi}{2} \Longrightarrow$