# Lecture 22: Asymptotic behaviour. Markov chains

**Aim lecture** Look at asymptotic behaviour of dynamical systems via e-value theory.

## Asymptotic behaviour

**e.g. 1** see MATLAB

Consider dts $\mathbf{x}(k+1) = A\,\mathbf{x}(k)$

where $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

$x_1(k) =$ popn of roadrunners

$x_2(k) =$ popn of

Describe behaviour as $k \longrightarrow \infty$

**A** MATLAB says

e-values are

e-vectors are

so $\mathbf{x}(k) =$

for some scalars $\alpha_1, \alpha_2$.

Let's assume $\alpha_1 \neq 0$. If $\alpha_1, \alpha_2$ are "chosen" randomly then we expect this to occur. We say more precisely, it occurs generically i.e. with probability 1.

This means dominant term is $\alpha_1 3^k \binom{2}{1}$.

$\therefore$ As $k \longrightarrow \infty$ expect

1) Popn growth: popn tends to

e.g. $\lim_{k \longrightarrow \infty} \frac{x_1(k+1)}{x_1(k)}$

$= \lim_{k \longrightarrow \infty} \frac{2\alpha_1 3^{k+1} + \alpha_2 2^{k+1}}{2\alpha_1 3^k + \alpha_2 2^k}$

$=$

Sim $\lim_{k \longrightarrow \infty} \frac{x_2(k+1)}{x_2(k)}$

2) Popn ratio $x_1(k) : x_2(k) \longrightarrow$

Check: $\lim_{k \longrightarrow \infty} \frac{2\alpha_1 3^k + \alpha_2 2^k}{\alpha_1 3^k + \alpha_2 2^k}$

**Upshot** Largest e-value gives expected popn growth rate,

corresp e-vector gives asymptotic popn distribution.

**Defn 1** The dts $\mathbf{x}(k+1) = A\,\mathbf{x}(k)$

is stable if for any soln $\mathbf{x}(k)$ we have

i.e. (assuming $A$ diag) all e-values of $A$ have magnitude

We say the dts is unstable if there is a soln $\mathbf{x}(k)$ with

The system in e.g. 1 above is unstable.

## Markov processes

In e.g. 1, total popn grows. In some applicns, total popn fixed but individuals "shuffle" around.

**e.g. 2** Residents of Jiggallong only vote labour or liberal.

$x_1(k)$ = no. labour voters on day $k$

$x_2(k)$ = no. liberal voters on day $k$

Suppose each day, after the nightly news, quarter of the labour voters switch to liberal

& half liberal voters switch to labour.

i.e. $x_1(k+1) =$

$x_2(k+1) =$

or $\mathbf{x}(k+1) = A\,\mathbf{x}(k)$

where $A =$

**N.B.** Entries in 1st (resp 2nd) column of $A$ is fraction of labour (resp liberal) voters voting labour/liberal the next day.

$\Longrightarrow$ total in each column is 1.

Suggests

**Defn 2** A dts $\mathbf{x}(k+1) = A\,\mathbf{x}(k)$ is said to be a Markov chain if

i) entries of $A$ are non-negative,

& ii) the sum of entries in each column is 1.

e.g. 3 above is a Markov chain.

$A$ is called a transition matrix.

**Prop 1** Suppose $A \in M_{nn}(\mathbb{R})$ has non-negative entries.

Then $(*) \quad \mathbf{x}(k+1) = A\,\mathbf{x}(k)$ describes a Markov chain iff $(1, 1, \ldots, 1)^T$ is an e-vector for $A^T$ with e-value 1.

**e.g.3 again** $A^T \begin{pmatrix} 1 \\ 1 \end{pmatrix} =$

Proof propn 1: Let $\mathbf{a}_1^T, \ldots, \mathbf{a}_n^T$ be the rows of

$A =$

By defn, (*) defines a Markov chain

$\iff$ the sum of the rows of $A$,

$\sum_i \mathbf{a}_i^T = (1, 1, \ldots, 1)$

$\iff$ sum of col of $A^T =$

$\iff$ $A^T$

$\iff$ $A^T$ has e-vector

**Prop 2** 1) $A \in M_{nn}(\mathbb{R})$ & $A^T$ have the same e-values.

2) If $\mathbf{x}(k+1) = A\,\mathbf{x}(k)$ defines a Markov chain, then 1 is an e-value for $A$.

Proof: 1) It suffices the prove the char poly for $A^T$ is the same as for $A$.

$\det(A^T - \lambda I) = \det$

2) follows from 1) & prop 1.

**e.g. 3 yet again** $A - I =$

$A - I$ is not invertible $\because$ rows of $A - I$ are parallel.

$\therefore$ 1 is indeed an e-value of $A$.

**Defn 3** Let $\mathbf{x}(k+1) = A\,\mathbf{x}(k)$ define a Markov chain. We say it's regular if for some $r \geq 1$ all entries of $A^r$ are positive.

**Thm** Let $\mathbf{x}(k+1) = A\,\mathbf{x}(k)$

define a regular Markov chain. Then

1) Any e-value $\lambda$ other than 1 satisfies $|\lambda| < 1$

2) The $\lambda = 1$ e-space is one-dimensional.

**Rem 1** In this case, as $k \longrightarrow \infty$

expect $\mathbf{x}(k) \longrightarrow$

Proof: None but see next section.

**Back to e.g. 3**

Recall $A = \begin{pmatrix} .75 & .5 \\ .25 & .5 \end{pmatrix}$

If total popn of Jiggallong is 6000, find asymptotic voting pattern.

Ans: Thm & prop 2 $\implies$ $\lambda = 1$ is an e-value. Let $\lambda = \mu$ be other e-value.

$\lambda = 1$ e-space:

$\ker(A - I) =$

Let $\mathbf{f}_1 =$

$\mathbf{f}_2$ be an e-vector with e-value $\mu$.

**Note** total popn $x_1(k) + x_2(k)$ remains constant $= 6000$.

Now $\mathbf{x}(k) = \alpha_1 1^k \, \mathbf{f}_1 + \alpha_2 \mu^k \, \mathbf{f}_2 = \alpha_1 \binom{2}{1} + \alpha_2 \mu^k \, \mathbf{f}_2$

for some scalars $\alpha_1, \alpha_2$.

$\lim_{k \longrightarrow \infty} \mathbf{x}(k) = \alpha_1 \binom{2}{1}$

Can't have $\alpha_1 = 0$ otherwise $\mathbf{x}(k) \longrightarrow \mathbf{0}$ as $k \longrightarrow \infty$ so total popn $\longrightarrow 0$ too, a contradiction.

Note $\implies 2\alpha_1 + \alpha_1 = \lim_{k \longrightarrow \infty} x_1(k) + x_2(k) = 6000$.

$\therefore \alpha_1 =$

Eventually, expect $\mathbf{x}(k)$ to tend to

$2000 \binom{2}{1} = \binom{4000}{2000}$ as $k \longrightarrow \infty$.

**Rem 2** a) If initial voting pattern is

$\mathbf{x}(0) = 2000 \binom{2}{1}$

then for all $k$ we have $\mathbf{x}(k) =$

since this is an e-vector for $A$ with e-value 1.

b) $\mathbf{x} = \binom{4000}{2000}$ is called an "equilibrium" of the dts for this reason.

## Some proofs

**Special Case of Thm** Let $\mathbf{x}(k+1) = A\,\mathbf{x}(k)$ define a Markov chain where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \& \quad a, b, c, d > 0$$

i) Then $A$ has e-values 1 & $\lambda = \det A$

ii) $|\lambda| < 1.$

Proof. i) Char poly is

$$p(\lambda) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix}$$

$=$

$\therefore$ product of e-values is $\det A$.

We know one e-value is 1 so

ii) holds $\because a, b, c, d \in (0, 1)$.