Lecture 21: Powers of matrices & discrete time systems

Aim lecture See applications of diagonalisation to computing powers of matrices & discrete time systems.

Geometric interpretation of powers

E.g. 1 Consider $A \in M_{22}(\mathbb{R})$ & assoc lin map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with

e-basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ & corresp e-values λ_1, λ_2.

$T^2\,\mathbf{v}_1 =$

$T^3\,\mathbf{v}_1 =$

$T^k\,\mathbf{v}_1 =$

It's thus easy to compute powers of T wrt B.

Matrix reprn of T^k (wrt B) is

$([T^k\,\mathbf{v}_1]_B \;\; [T^k\,\mathbf{v}_2]_B)$

$= ([\lambda_1^k\,\mathbf{v}_1]_B \;\; [\lambda_2^k\,\mathbf{v}_2]_B) =$

Suggests diagonalisation useful for computing

Powers of matrices

The algebraic reason why diagonal matrices are nice is

Lemma The product of diagonal matrices is

$$\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_n \end{pmatrix} = \begin{pmatrix} \lambda_1\mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n\mu_n \end{pmatrix}$$

Proof: Easy computation.

Prop 1) Consider the diagonal matrix

$D =$

Then $D^k =$

2) If $A = MDM^{-1}$ then

$A^k = MD^kM^{-1} =$

Proof: 1) by

2) $A^k =$

$=$

e.g. 2 a) Verify diagn

$$A = \begin{pmatrix} .8 & .4 \\ .2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} .6 & 0 \\ 0 & 1.2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

b) Find A^k.

A a) $MD =$

$AM =$

so $AM = MD$ or equiv, $A = MDM^{-1}$.

b) $A^k = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} .6^k & 0 \\ 0 & 1.2^k \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 2 \times .6^k & 1.2^k \\ -.6^k & 1.2^k \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$

$=$

Example of decoupled dts

e.g. 3 Defence budget. Let

$x_1(k) =$ defence expenditure of hobbits in year k.

$x_2(k) =$ defence expenditure of

Suppose popn separate so year to year change in expenditure governed by recursion reln

$x_1(k+1) = .8x_1(k)$

$x_2(k+1) = .6x_2(k)$

Soln is

Can rewrite 2 recurrence reln as single reln

$\mathbf{x}(k+1) =$

where $\mathbf{x}(k) = \binom{x_1(k)}{x_2(k)}$.

Q What if A is not diagonal?

Diagonalisation & decoupling dts

Let $\mathbf{x}(k) =$

Consider the following recurrence reln

$$(*) \qquad \mathbf{x}(k+1) = A\,\mathbf{x}(k)$$

where $A \in M_{nn}(\mathbb{R})$.

This is another example of a discrete time system.

Note $\mathbf{x}(1) =$

$\mathbf{x}(2) =$

$\mathbf{x}(k) =$

Conclusion Can solve (*) by computing powers of matrices as in e.g. 2.

Alternatively, repeat analysis for cts to get

Thm Suppose $A = MDM^{-1}$

where $M = (\mathbf{f}_1 \ldots \mathbf{f}_n)$

$D =$

The soln to dts $\mathbf{x}(k+1) = A\,\mathbf{x}(k)$ is

(†) $\mathbf{x}(k) = \alpha_1 \lambda_1^k \mathbf{f}_1 + \ldots + \alpha_n \lambda_n^k \mathbf{f}_n.$

for some scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$.

Proof: As in lect 20 or directly by induction on k.

$k = 0$: Pick $\alpha_1, \ldots, \alpha_n$ scalars with

$\mathbf{x}(0) = \alpha_1 \mathbf{f}_1 + \ldots + \alpha_n \mathbf{f}_n$

which is possible since $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is

$k > 0$: Suppose (†) holds for some value of k.

$\mathbf{x}(k+1) = A\,\mathbf{x}(k)$

$= A(\alpha_1 \lambda_1^k \mathbf{f}_1 + \ldots + \alpha_n \lambda_n^k \mathbf{f}_n)$

$= \alpha_1 \lambda_1^k A \mathbf{f}_1 + \ldots + \alpha_n \lambda_n^k A \mathbf{f}_n$

$=$

Thm is proved.

Rem Compare with soln in cts case:

Example of dts

e.g. 3 again Arms race. Let's put hobbit & orc popn together so defence expenditure now evolves via

$$x_1(k+1) = .8x_1(k) + .4x_2(k)$$
$$x_2(k+1) = .2x_1(k) + x_2(k)$$

N.B. Expenditure for hobbits depends positively on

Need solve $\mathbf{x}(k+1) = \begin{pmatrix} .8 & .4 \\ .2 & 1 \end{pmatrix} \mathbf{x}(k)$

A Same matrix A as in e.g. 2 so $A = MDM^{-1}$ where

$$D = \begin{pmatrix} .6 & 0 \\ 0 & 1.2 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

Method 1: Thm $\Longrightarrow$

$\mathbf{x}(k) =$

for some scalars α_1, α_2.

Method 2: Use formula in e.g. 2 for A^k. Ex. try this at home.

E.g. 3 cont'd Find expenditure in k-th year if initially hobbits spend 1 gold piece and orcs 0.

A We solve

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{x}(0) =$$

$\therefore \alpha_1 =$

$\therefore \mathbf{x}(k) =$