Lecture 20: DEs and diagonalisation

Aim lecture See how diagonalisation is useful for solving differential eqns.

Motivation

e.g. 1 $y_1(t) =$ popn of hobbits in

$y_2(t) =$ popn of orcs

If 2 popn kept separate as here then popn growth governed by a pair of DEs which typically looks something like:

$$\begin{aligned} y_1'(t) &= 3y_1(t) \\ y_2'(t) &= 2y_2(t) \end{aligned} \qquad (*)$$

Soln: Easy, solve 2 eqns <u>separately</u>

$y_1(t) =$

Suppose now we put the two popns together in

Typical DEs describing popn growth is

$$\begin{aligned} y_1'(t) &= 3y_1(t) - 2y_2(t) \\ y_2'(t) &= -y_1(t) + 2y_2(t) \end{aligned} \qquad (\dagger)$$

These are "coupled" DEs i.e. y_1', y_2' each depend on both $y_1 \& y_2$. We'll use diag to

N.B. Growth rate of hobbit popn depends positively on hobbit popn &

Notn $\mathbf{y}(t) =$

$\mathbf{y}'(t) = \frac{d\,\mathbf{y}}{dt} :=$

In e.g. 1, we can write

$$\begin{pmatrix} 3y_1 - 2y_2 \\ -y_1 + 2y_2 \end{pmatrix} =$$

so $\mathbf{y}'(t) = A\,\mathbf{y}(t)$ where $A = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$.

Note In decoupled case (*) above, still have $\mathbf{y}'(t) = A\,\mathbf{y}(t)$ but now

$A =$

Diagonalisation & decoupling DEs

Consider more generally

$\mathbf{y}(t) =$

& system of n linear DEs

$\mathbf{y}'(t) = A\,\mathbf{y}(t)$

where $A \in M_{n,n}(\mathbb{R})$.

Lemma For $C \in M_{n,n}(\mathbb{R})$

$\frac{d}{dt}\big(C\,\mathbf{y}\big) = C\frac{d\,\mathbf{y}}{dt}$.

Proof: Clear from case $n = 2$. Suppose

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$\frac{d}{dt}(C\,\mathbf{y}) =$

To solve $\mathbf{y}'(t) = A\,\mathbf{y}(t)$, suppose we can diag $A = MDM^{-1}$ with $M = (\mathbf{f}_1 \ldots \mathbf{f}_n)$

& $D =$

Thm 1) If we change variables to

$\mathbf{x}(t) = M^{-1}\,\mathbf{y}(t)$

then get decoupled eqn

(*) $\qquad \frac{d\,\mathbf{x}}{dt} = D\,\mathbf{x}$

2) Soln to (*) is

$x_i(t) = \alpha_i e^{\lambda_i t}$ for $i = 1, \ldots, n$

& scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$.

3) Soln to original DE $\mathbf{y}'(t) = A\,\mathbf{y}(t)$ is

$\mathbf{y}(t) = M\,\mathbf{x}(t) = \alpha_1 e^{\lambda_1 t}\,\mathbf{f}_1 + \ldots + \alpha_n e^{\lambda_n t}\,\mathbf{f}_n$

where $\mathbf{f}_1, \ldots, \mathbf{f}_n$ are e-vectors with corresp e-values $\lambda_1, \ldots, \lambda_n$.

Proof: 1) $\mathbf{y}'(t) = A\,\mathbf{y} = MDM^{-1}\,\mathbf{y} = MD\,\mathbf{x}$.

Also, lemma $\implies \frac{d}{dt}(M\,\mathbf{x}(t)) =$

Equating & noting M invertible we see $\frac{d\,\mathbf{x}}{dt} = D\,\mathbf{x}$.

2) (*) corresponds to system of linear DEs

cont'd

3) Just multiply matrices.

$\mathbf{y}(t) = M\,\mathbf{x}(t) = (\mathbf{f}_1 \ldots \mathbf{f}_n)$

$= \alpha_1 e^{\lambda_1 t}\,\mathbf{f}_1 + \ldots + \alpha_n e^{\lambda_n t}\,\mathbf{f}_n$

Example

e.g. 1 completed

We diag A

$$0 = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 \\ -1 & 2-\lambda \end{vmatrix}$$

The e-values are 4,1.

E-vectors?

$\lambda = 4 : \ker(A - \lambda I) =$

An e-vector is

$\lambda = 1 : \ker(A - \lambda I) =$

An e-vector is

Thm 3) $\implies$

$\mathbf{y}(t) =$

for some scalars α_1, α_2.

i.e. $y_1(t) =$

$y_2(t) =$

e.g. 2 Suppose in e.g. 1 that initial popn is $\mathbf{y}(0)^T = (4000, 1000)$. Solve the IVP.

Ans: We need only solve for α_1, α_2.

From Gaussian elim or guessing see

$\alpha_1 =$

The soln is thus

$\mathbf{y}(t) =$

e.g. 3 What happens in e.g. 2 as $t \longrightarrow \infty$?

Nasty hobbits!

N.B. Key to limiting behaviour is e-value of max magnitude.

Second order DEs

We can convert any 2nd order const coeff linear ODE into a pair of linear ODEs in 2 var as in following

E.g. 4 Solve IVP

$$y'' - 3y' + 2y = 0 \quad , \quad y(0) = 2, y'(0) = 3$$

Ans: Let $y_1 = y, y_2 = y'$

$y_1' = y' = y_2$

$y_2' = y'' = -2y + 3y' = -2y_1 + 3y_2$

i.e. $\mathbf{y}' =$

Diag $A =$

$\det(A - \lambda I) =$

Hence e-values are 2,1.

E-vectors:

$\lambda = 2 : \ker(A - \lambda I) =$

An e-vector is

$\lambda = 1 : \ker(A - \lambda I) =$

An e-vector is

Hence, (from thm 3)) general soln is

$\mathbf{y}(t) =$

Need now find integration constants.