Lecture 2: Computing Powers. Extracting roots.

Aim Lecture Find simple methods to compute powers & extract roots using complex exponential fn.

Euler's formula for complex exp fn

Lemma $(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$
$= \cos(\theta+\phi) + i\sin(\theta+\phi)$.

Proof LHS $= (\cos\theta\cos\phi - \sin\theta\sin\phi)$
$+i(\sin\theta\cos\phi + \cos\theta\sin\phi)$

Euler's Formula For $\theta \in \mathbb{R}$
$e^{i\theta} := \cos\theta + i\sin\theta$.

More gen for $a, b \in \mathbb{R}$
$e^{a+bi} :=$

N.B. Comparing with polar form we see

$|e^{a+bi}| =$

$\operatorname{Arg} e^{a+bi}$

e.g. $e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} =$

Can also see this from Argand diagram

Properties of exponential fn

Q Why is Euler's defn sensible?

One **A** Have desirable

Facts 1. It recovers real exp fn when $z \in \mathbb{R}$.

2. $e^{z+z'} = e^z e^{z'}$

3. $(e^z)^{-1} = e^{-z}$

4. $\frac{e^z}{e^{z'}} = e^{z-z'}$

5. For $n \in \mathbb{Z}$, $(e^z)^n = e^{nz}$

Proof: 1. $e^{a+0i} = e^a(\cos 0 + i\sin 0) =$

2. Write $z = a + bi, z' = a' + b'i$

$\text{LHS} = e^{a+bi+a'+b'i} = e^{(a+a')+i(b+b')}$

$= e^{a+a'}(\cos(b + b') + i \sin(b + b'))$

$\overset{\text{lemma}}{=} e^a e^{a'}(\cos b + i \sin b)(\cos b' + i \sin b')$

$= e^z e^{z'}$.

3. $e^{-z}e^z = e^0 = 1$ so

4. Just use 2. & 3.

5. For $n \geq 1$ it follows by

Clear for $n = 0$.

For $n < 0$,

$(e^z)^n = \frac{1}{(e^z)^{-n}} = \frac{1}{e^{-nz}} = e^{nz}$.

An immediate corollary is

De Moivre's Thm

$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$

Proof: $\text{LHS} = (e^{i\theta})^n =$

Fact 1. For $n \in \mathbb{Z}$,

$e^{i(\theta+2n\pi)} = e^{i\theta}$

2. $e^{i\theta} = e^{i\theta'} \implies \theta - \theta' = 2n\pi$ for some $n \in \mathbb{Z}$.

Proof: 1. holds as cos, sin have period 2π.

2. $e^{i\theta} = e^{i\theta'} \implies e^{i(\theta-\theta')} = 1$

$\implies \cos(\theta - \theta') = 1$ so $\theta - \theta'$ is a multiple of 2π.

e.g. $e^{-i\pi/2} = e^{i3\pi/2}$ as can also been seen from

Products in polar form

For $r, \theta \in \mathbb{R}$,

$r(\cos\theta + i\sin\theta) = re^{i\theta}$

This is alternate polar form.

N.B. $re^{i\theta} = r'e^{i\theta'}$ iff $r = r'$ & $\frac{\theta-\theta'}{2\pi} \in \mathbb{Z}$.

Consequences For $z, w \in \mathbb{C}$

1. $|\frac{z}{w}| = \frac{|z|}{|w|}, |zw| = |z||w|$.

2. $\operatorname{Arg} zw = \operatorname{Arg} z + \operatorname{Arg} w + 2n\pi$

for some $n \in \mathbb{Z}$.

$\operatorname{Arg} \frac{z}{w} =$

3. $|z^n| =$

4. $\operatorname{Arg} z^n =$

Proof half of 1 & 2 only. Others sim.

Let $z = re^{i\theta}, w = se^{i\phi}$

$\frac{z}{w} =$

This is the polar form for $\frac{z}{w}$

Hence, $|\frac{z}{w}| =$

$\operatorname{Arg} \frac{z}{w}$

e.g. Let $z = -1 + i$ so $\operatorname{Arg} z =$

Then $z^2 =$

so $\operatorname{Arg} z^2 =$

Note $2 \operatorname{Arg} z = \operatorname{Arg} z^2 + 2\pi$.

e.g. Let $z = -1 + i, w = 3e^{-2i}$. We find the polar form of zw.

$|zw| =$

$\operatorname{Arg} zw$

Powers via polar form

e.g. 1 Find $(\sqrt{3} - i)^{100}$

Dumb method

Good Method: Write $z = \sqrt{3} - i$ in polar form

$|z| =$

$\operatorname{Arg} z =$

$\therefore z = 2e^{i\pi/6}$

$z^{100} = 2^{100} e^{i100\pi/6} = 2^{100} e^{i50\pi/3}$

$= 2^{100} e^{2\pi i/3}$

as $e^{48\pi i/3} = e^{16\pi i} =$

$= 2^{100}($

$= -2^{99}$

n-th roots via polar form

e.g. 2 Find all cube roots of $8i$.

A Suppose $z^3 = 8i, z = re^{i\theta}$

Polar form of $8i =$

$z^3 = r^3 e^{i3\theta}$

Equate moduli:

Equate arg: $3\theta = \frac{\pi}{2} + 2n\pi$

for some $n \in \mathbb{Z}$.

N.B. $-\pi < \theta \leq \pi \Longrightarrow -3\pi < 3\theta \leq 3\pi$

Hence, $3\theta =$

$\therefore \theta = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

So the three cube roots are

$z = re^{i\theta}$

$=$

Let's plot the 3 cube roots

Roots of unity

Method in e.g. 2 gives

Thm 1 The n-th roots of unity i.e. solns to $z^n = 1$ are

$$z = 1, e^{2\pi i/n}, e^{4\pi i/n}, \ldots, e^{2\pi i(n-1)/n}$$

Check: For $k \in \mathbb{Z}$, $z = e^{i2k\pi/n} \Longrightarrow$

$z^n =$

Thm 2 Let $\omega := e^{2\pi i/n}$,

If z_0 is an n-th root of $\alpha \in \mathbb{C}$ then the n-th roots of α are

$z = z_0,$

Proof: $z_0\omega^j$ is a root $\because$

$(z_0\omega^j)^n =$

Conversely, if $z_1^n = \alpha$

$(\frac{z_1}{z_0})^n$

Thm 1 $\Longrightarrow$

so $z_1 =$

Multn & rotation

Shows multn by $e^{i\phi}$ rotates anti-clockwise by ϕ.

Gives geom interpretation of thm 2. If z_0 is an n-th root of α then the other roots are obtained by rotating $\frac{2\pi}{n}, \frac{4\pi}{n}, \ldots$.

n-th roots are equally spaced around a circle.

Square roots via cartesian form

e.g. 3 Solve $z^2 = -5 + 12i$

A Let $z = a + bi, a, b \in \mathbb{R}$

$z^2 = (a^2 - b^2) + 2abi$

Equate real & imag parts

Solve simultaneously by guessing or elim a using

$a^2 + b^2 = |z|^2 = |z^2| = |-5 + 12i| = \sqrt{5^2 + 12^2} = 13$

$\therefore 2a^2 =$

$b =$

So square roots are $z = a + bi = \pm(2 + 3i)$.

Quadratic formula

$$az^2 + bz + c = 0, \quad a, b, c \in \mathbb{C}$$

has solns

e.g. 4 Solve $z^2 + (-4+i)z + (5-5i) = 0$.

discriminant $= b^2 - 4ac = (-4+i)^2 - 4(5-5i)$

$= 16 - 8i + i^2 - 20 + 20i = -5 + 12i$

From e.g. 3 we see

$z = \frac{1}{2}(4 - i \pm (2+3i))$

$=$

$= 3 + i$ or $1 - 2i$