Lecture 19: Eigenbases. Diagonalisation

Aim (this lecture): See how matrices with nice basis consisting of e-vectors can be diagonalised.

Aim (next few lectures): See how diagonalised forms simplify computations in various contexts.

Eigenbases

For a linear transformation $T : V \longrightarrow V$, our preferred basis is given by

Defn An eigenbasis for T is a basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ such that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are e-vectors for T.

We sim define the eigenbasis for a square matrix.

e.g. 1 Recall lect 18 e.g. 2, a diag matrix $D \in M_{nn}$ has e-basis $B =$

Corresponding e-values are

e.g. 2 Recall lect 18 e.g. 5

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$$

has e-basis $B = \{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)\}$

Let's check this independently.

B is lin indep $\because$

Also $\dim \mathbb{R}^2 = 2$ so B is a basis for $\mathbb{R}^n$.

Let's check the vectors are e-vectors.

$A\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$

$\implies \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ is an e-vector with e-value

$A\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$

$\implies \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$

Q Why is an e-basis nice? One answer is

Thm 1 Let $T : V \longrightarrow V$ be a lin map

$\& B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an e-basis with corresponding e-values $\lambda_1, \ldots, \lambda_n$.

The matrix D representing T wrt B is

Proof: Gen matrix reprn thm lect 16 $\Longrightarrow$

$D = ([T\,\mathbf{v}_1]_B \ldots [T\,\mathbf{v}_n]_B)$

$= ([\lambda_1\,\mathbf{v}_1]_B \ldots [\lambda_n\,\mathbf{v}_n]_B)$

$=$

Diagonalising matrices

Thm-Defn $A \in M_{n,n}(\mathbb{F})$ is diagonalisable (over $\mathbb{F}$) if there's a diag matrix

$D =$

& an invertible $n \times n$-matrix $M = (\mathbf{v}_1\, \mathbf{v}_2 \ldots \mathbf{v}_n)$ s.t.

$A = MDM^{-1}$.

Finding such an expression is called diagonalising A.

In this case $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an e-basis for A

& the corresp e-values are

Proof: M invertible $\implies$ columns $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of M form a basis for $\mathbb{F}^n$.

Let's now check $\mathbf{v}_i$ are e-vectors with e-values λ_i.

$AM = (A\,\mathbf{v}_1 \ldots A\,\mathbf{v}_n)$

$MD = (\mathbf{v}_1 \ldots \mathbf{v}_n)$

$= (\lambda_1\, \mathbf{v}_1 \ldots \lambda_n\, \mathbf{v}_n)$.

But $A = MDM^{-1} \implies AM = MD$ so comparing columns of both sides we see $A\,\mathbf{v}_i = \lambda_i\, \mathbf{v}_i$.

Proof of thm-defn is now complete.

Reading proof backwards gives converse

Thm 2 Suppose $A \in M_{n,n}(F)$ has e-basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ & corresp e-values $\lambda_1, \ldots, \lambda_n$.

Then A is diagonalisable with $A = MDM^{-1}$ where

i) $M = ($

& ii) $D =$

Proof: B a basis $\implies$

Since $A\,\mathbf{v}_i = \lambda_i\,\mathbf{v}_i$,

above computation shows $AM = MD$

$\therefore A = MDM^{-1}$ & A is diagonalisable.

Example of diagonalisation

e.g. 2 again Diagonalise

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}.$$

A Saw we have e-basis $B = \{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)\}$ with corresp e-values $2, -1$. Hence

$A = MDM^{-1} =$

e.g. 3 Diagonalise

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -4 & 6 \\ 0 & -3 & 5 \end{pmatrix}$$

Note, if A as in e.g. 2, then $C =$

$$\det(C - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & -4-\lambda & 6 \\ 0 & -3 & 5-\lambda \end{vmatrix}$$

$= (1-\lambda)\det(A - \lambda I)$

$= (1-\lambda)(\lambda^2 - \lambda - 2)$

$=$

So e-values are

E-vectors:

$\lambda = 1$ e-space:

$$\ker(C - \lambda I) = \ker \begin{pmatrix} & 1 & 0 \\ 0 & & 6 \\ 0 & -3 & \end{pmatrix}$$

An e-vector with e-value 1 is

$\lambda = 2$ e-space:

$$\ker(C - \lambda I) = \ker \begin{pmatrix} & 1 & 0 \\ 0 & & 6 \\ 0 & -3 & \end{pmatrix}$$

An e-vector with e-value 2 is

$\lambda = -1$ e-space:

$$\ker(C - \lambda I) = \ker \begin{pmatrix} & 1 & 0 \\ 0 & & 6 \\ 0 & -3 & \end{pmatrix}$$

An e-vector with e-value -1 is

$\therefore$ an e-basis for A is

$B =$

N.B. You can check lin indep of B directly, so B is a basis. Can also see B is a basis by using prop at end of lecture.

If $M =$

then $C =$

N.B. Diagonalisation is NOT unique.

Can swap

Can scale

Existence of e-bases

One of the few results we prove in this course guaranteeing existence of e-bases is

Prop Let $T : V \longrightarrow V$ be a lin map.

1) If $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ are e-vectors with distinct e-values then they are lin indep.

2) If $A \in M_{n,n}(\mathbb{F})$ has e-vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ whose e-values are distinct, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an e-basis for A.

Proof: Since $\dim \mathbb{F}^n = n$, 1) $\implies$ 2).

Let's prove 1) by induction on m. $m = 1$ case is clear since e-vectors are non-zero.

First observe

$(T - \lambda_m \,\mathrm{id})\,\mathbf{v}_i =$

Consider a linear relation

$$(*) \qquad \alpha_1\,\mathbf{v}_1 + \ldots + \alpha_m\,\mathbf{v}_m = \mathbf{0}.$$

for some α_i

Applying lin map $T - \lambda\,\mathrm{id}$

$$\mathbf{0} = (T - \lambda_m\,\mathrm{id})(\alpha_1\,\mathbf{v}_1 + \ldots + \alpha_m\,\mathbf{v}_m)$$

$$= \alpha_1(\lambda_1 - \lambda_m)\,\mathbf{v}_1 + \ldots + \alpha_{m-1}(\lambda_{m-1} - \lambda_m)\,\mathbf{v}_{m-1}$$
$$+\alpha_m(\lambda_m - \lambda_m)\,\mathbf{v}_m$$

which is a lin combn of $\{\mathbf{v}_1, \ldots, \mathbf{v}_{m-1}\}$ only.

Since by induction $\{\mathbf{v}_1, \ldots, \mathbf{v}_{m-1}\}$ is lin indep, we have

$0 = \alpha_1(\lambda_1 - \lambda_m) =$

Since the e-values are distinct, we have

$0 =$

The relation (*) reduces to $\alpha_m\,\mathbf{v}_m = \mathbf{0}$ so also

$\alpha_m = 0$.

This proves the prop.

Real symmetric matrices

Recall a (square) matrix A is symmetric if $A = A^T$.

Fact Let $A \in M_{nn}(\mathbb{R})$ be a real symmetric matrix. Then

a) The e-values of A are all real.

b) There is an e-basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ which is o/n.

No proof.

e.g. 4 Techniques of this lecture can be used to diaonalise the real symmetric matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}.$$

Corresp e-basis $B = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$ is not o/n

$\because$

We can obtain o/n e-basis by scaling