Lecture 18: Eigenvalues & eigenvectors

Aim Lecture Introduce concept of eigenvalues & eigenvectors which give natural bases.

Philosophy of eigenvectors

From now on study lin maps of form $T : V \longrightarrow V$

i.e. where domain =

Key Point Linear $T : V \longrightarrow V$ often pick out their own preferred coord

Motivational example

e.g. 1 $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ reflection about line L containing $\mathbf{0}$.

N.B. One can shows geom, that T is linear.

Q What co-ord system is most natural?

A Natural choice is to

i) let one co-ord axis be

ii) other

N.B. This corresp to basis $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ where $\mathbf{u}_1$ spans

Q If this T given by matrix, then how do you find this basis algebraically from the matrix?

A $T\,\mathbf{u}_1 =$ $\qquad$ $T\,\mathbf{u}_2$

so T sends $\mathbf{u}_1, \mathbf{u}_2$ to scalar

One nice consequence of this is the matrix representing T wrt $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ is

$([T\,\mathbf{u}_1]_B \;\; [T\,\mathbf{u}_2]_B) = ([\mathbf{u}_1]_B \;\; [-\,\mathbf{u}_2]_B)$

$=$

This suggests

Eigenvectors/values

Key to finding preferred basis is

Defn 1 Let $T : V \longrightarrow V$ be a linear map,

$\mathbf{v} \in V - \mathbf{0}, \lambda$ a scalar

s.t. $T\,\mathbf{v} = \lambda\,\mathbf{v}$.

Then we call λ an eigenvalue of T

& $\mathbf{v}$ an eigenvector of T with

If $A \in M_{nn}(\mathbb{F})$ (square!), the e-values/e-vectors of A are the e-values/e-vectors of its associated linear map $\mathbb{F}^n \longrightarrow \mathbb{F}^n$.

i.e. $A\,\mathbf{v} = \lambda\,\mathbf{v}$ means $\mathbf{v} \in \mathbb{F}^n$ is an

e.g. 1 again $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is reflection about line Span $\mathbf{u}_1$.

$T\,\mathbf{u}_1 = 1\,\mathbf{u}_1 \implies$

For non-zero $\mathbf{u}_2 \perp \mathbf{u}_1$

e.g. 2 Diagonal matrices i.e. of form

$D =$

Then $D\,\mathbf{e}_i =$

so $\mathbf{e}_i$ is an e-vector with e-value

N.B. Here we have a basis of e-vectors, viz. the standard basis.

This is a good basis in this case $\because$ in the fn of $\mathbf{x}$ defined by

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = D\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{pmatrix},$$

y_i is a fn of

e.g. 3 $V = \mathcal{C}^\infty$ is vect space of infinitely differentiable functions on $\mathbb{R}$.

Let $T : V \longrightarrow V$ be differentiation.

Then

Eigenspaces

Re-interpret e-vectors using

Lemma Let $T : V \longrightarrow V$ be a linear map.

$\mathbf{v} \in V - \mathbf{0}$ is an e-vector with e-value λ iff $\mathbf{v} \in \ker(T - \lambda\,\mathrm{id})$.

Proof: Let $\mathbf{v} \neq \mathbf{0}$. It is an e-vector of T with e-value λ

$\iff T\,\mathbf{v} =$

$\iff T\,\mathbf{v} - \lambda\,\mathbf{v} = \mathbf{0}$

$\iff (T - \lambda\,\mathrm{id})$

$\iff \mathbf{v} \in$

Rem 1) Sim $\mathbf{v} \neq \mathbf{0}$ is an e-vector of a square matrix A with e-value λ iff

2) Lemma $\implies$ if you know the e-values of T then the e-vectors can be found by computing the kernel of some lin map.

Defn 2 The λ-eigenspace of a lin map $T : V \longrightarrow V$ is the subspace $\ker(T - \lambda\,\mathrm{id})$. In other words, it is the set of e-vectors with e-value λ with the vector

e.g. 4 Let

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

a) Show 3 is not an e-value of D.

b) Find the $\lambda = 2$ e-space of D.

A a) $D - 3I =$

which is invertible so $\ker(D - 3I) =$

Lemma $\implies$ D has no non-zero e-vector with e-value 3.

$\therefore$ 3 is

b) $D - 2I =$

$\therefore \lambda = 2$ e-space is

$\ker(D - 2I) =$

$= \mathrm{Span}(\mathbf{e}_2, \mathbf{e}_3)$

The e-vectors of D with e-value 2 are the non-zero vectors in the plane $\mathrm{Span}(\mathbf{e}_2, \mathbf{e}_3)$.

Finding e-values. Characteristic polynomials

Find e-values using

Prop-Defn Let $A \in M_{n,n}(\mathbb{F})$. The fn of λ

$p(\lambda) := \det(A - \lambda I)$

is called the characteristic polynomial of A.

1) $p(\lambda)$ is a poly of degree n

2) Its roots are e-values of A.

Proof: 1) Will be clear from any example (see below).

2) λ is an e-value of A

iff A has an e-vector with e-value λ

$\overset{\text{lemma}}{\Longleftrightarrow}$ $\ker(A - \lambda I)$ has a non-zero vector in it

iff $\ker(A - \lambda I) \neq$

iff $A - \lambda I$ is not

iff det

N.B. To ensure $p(\lambda)$ has roots, often advantageous to work with $\mathbb{F} = \mathbb{C}$.

Example of finding e-values/vectors

e.g. 5 Find e-values & e-vectors of

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$$

Ans: Step 1: Find e-values by finding roots of

$p(\lambda) = \det(A - \lambda I)$

$=$

$= (\lambda - 2)(\lambda + 1)$

$\therefore$ e-values are $2, -1$.

Step 2: Find corresponding e-vectors

For $\lambda = 2$ e-space:

$\ker(A - \lambda I) = \ker$

$\therefore$ e-vectors of A with e-value 2 are those of form

For $\lambda = -1$ e-space:

$\ker(A - \lambda I) =$

$\therefore$ e-vectors of A with e-value -1 are those of form

e.g. 6 We find the e-values of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

A The char poly is

$\det(A - \lambda I) =$

so e-values are

Upshot of this example is: any upper triangular matrix $A = (a_{ij})$ i.e. where $a_{ij} = 0$ whenever $j < i$, has e-values the diagonal entries.