Lecture 17: Linear Eqns in Abstract Vector Spaces

Aim Lecture Understand in what sense an eqn such as $\frac{dy}{dx} + x^2 y = e^x$ is linear. Apply theory of kernels & images to such eqns.

Link with linear ODEs

e.g. 1 Multn by a fn is linear.

For $g(x) \in \mathcal{R}[\mathbb{R}]$, define $T : \mathcal{R}[\mathbb{R}] \longrightarrow \mathcal{R}[\mathbb{R}]$ by $(Tf)(x) = g(x)f(x)$.

Then T is linear. Why? For $f_1, f_2 \in \mathcal{R}[\mathbb{R}], \lambda \in \mathbb{R}$,

Addn condn: $(T(f_1+f_2))(x) = g(x)(f_1(x)+f_2(x))$

$= g(x)f_1(x) + g(x)f_2(x) =$

$(Tf_1)(x) + (Tf_2)(x) = (Tf_1 + Tf_2)(x)$

so addn condn holds.

Scalar multn cond: $(T(\lambda f_1))(x) =$

$g(x)\lambda f_1(x) = (\lambda(Tf_1))(x)$

so scalar multn condn holds too & T is linear.

e.g. 2 Define $T : \mathbb{P}_3(\mathbb{R}) \longrightarrow \mathbb{P}_2(\mathbb{R})$ by

$$(Tp)(x) = (1+x)\frac{d^2p}{dx^2} - \frac{dp}{dx}$$

(Note $\deg Tp < \deg p$ so the co-domain is legitimate.) We show T is linear.

Why? For $p, q \in \mathbb{P}_3, \lambda \in \mathbb{R}$,

Addn condn: $(T(p+q))(x) =$

$= (1+x)(p+q)''(x) - (p+q)'(x)$

$=$

$=$

Scalar Multn condn: $(T(\lambda p))(x) =$

$(1+x)\lambda p''(x) - \lambda p'(x) =$

$\lambda[(1+x)p''(x) - p'(x)] = [\lambda(Tp)](x)$.

Sophisticated reason why T is linear: composition of lin maps differentiation & multn by fn is linear, so any linear combn of these such as T is linear too.

Generalised matrix representation theorem

To compute kernels & images for linear maps such as those in previous examples, need to represent them with matrices using

Thm (Generalised Matrix Rep Thm)

Let $T : V \longrightarrow W$ be a lin map,

$B_V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an ordered basis for V,

$B_W = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be an ordered basis for W.

Let $A = (\mathbf{a}_1 \ldots \mathbf{a}_n)$ be the $m \times n$-matrix

with columns $\mathbf{a}_i = [T\,\mathbf{v}_i]_{B_W}$.

Then $(*)$ $\qquad [T\,\mathbf{v}]_{B_W} = A[\mathbf{v}]_{B_V}$.

We say the matrix A represents T wrt $B_V \& B_W$.

Why? Let $\mathbf{x} = [\mathbf{v}]_{B_V}$ so

$\mathbf{v} = x_1\,\mathbf{v}_1 + \ldots + x_n\,\mathbf{v}_n$.

Recall T preserves lin combn so

$T\,\mathbf{v} = T(x_1\,\mathbf{v}_1 + \ldots + x_n\,\mathbf{v}_n)$

$= x_1 T\,\mathbf{v}_1 + \ldots + x_n T\,\mathbf{v}_n.$

Linearity of coords gives

$[T\,\mathbf{v}]_{B_W} = [x_1 T\,\mathbf{v}_1 + \ldots + x_n T\,\mathbf{v}_n]_{B_W}$

$= x_1 [T\,\mathbf{v}_1]_{B_W} + \ldots + x_n [T\,\mathbf{v}_n]_{B_W}$

$= x_1\,\mathbf{a}_1 + \ldots + x_n\,\mathbf{a}_n = A\,\mathbf{x} = A[\mathbf{v}]_{B_V}.$

e.g. 2 cont'd Recall $T : \mathbb{P}_3(\mathbb{R}) \longrightarrow \mathbb{P}_2(\mathbb{R})$ defined by

$$(Tp)(x) = (1+x)\frac{d^2p}{dx^2} - \frac{dp}{dx}.$$

Find the matrix reprn of T wrt bases

$B_3 = \{1, x, x^2, x^3\}$,

$B_2 = \{1, x, x^2\}$.

Ans: Thm $\Longrightarrow$ representing matrix given by

$A = ([T1]_{B_2}\ [Tx]_{B_2}\ [Tx^2]_{B_2}\ [Tx^3]_{B_2})$

$$T1 = (1+x)\frac{d^2}{dx^2}(1) - \frac{d}{dx}(1) = 0.$$

$$[T1]_{B_2} =$$

$$Tx = (1+x)\frac{d^2}{dx^2}(x) - \frac{d}{dx}(x) =$$

$$[Tx]_{B_2} =$$

$$Tx^2 =$$

$$=$$

$$[Tx^2]_{B_2} =$$

$$Tx^3 =$$

$$=$$

$$[Tx^3]_{B_2} =$$

Hence, representing matrix is

$A =$

For fun, let's check $[Tp]_{B_2} = A[p]_{B_2}$ when $p(x) = 2x + x^2$.

$[T(2x + x^2)]_{B_2} =$

$[(1 + x)\frac{d^2}{dx^2}(2x + x^2) - \frac{d}{dx}(2x + x^2)]_{B_2}$

$=$

$=$

$A[2x + x^2]_{B_3} = A(0, 2, 1, 0)^T$

$=$

So equality is checked.

Computing kernels & images

Often can use matrix reprn thm &

Prop Let $T : V \longrightarrow W$ be a lin map & B_V, B_W be finite ordered bases for V, W resp. Let A be the matrix representing T wrt B_V, B_W. Then

a) $\mathbf{v} \in \ker T$ iff $[\mathbf{v}]_{B_V} \in \ker A$.

b) In fact, $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ is a basis for $\ker T$ iff $\{[\mathbf{v}_1]_{B_V}, \ldots, [\mathbf{v}_s]_{B_V}\}$ is a basis for $\ker A$.

c) $\mathbf{w} \in \text{im } T$ iff $[\mathbf{w}]_{B_W} \in \text{im } A$.

d)In fact, $\{\mathbf{w}_1, \ldots, \mathbf{w}_r\}$ is a basis for im T iff $\{[\mathbf{w}_1]_{B_W}, \ldots, [\mathbf{w}_r]_{B_W}\}$ is a basis for im A.

Why? Recall $[T\,\mathbf{v}]_{B_W} = A[\mathbf{v}]_{B_V}$.

Check ($\Longrightarrow$). If $\mathbf{v} \in \ker T$ so $T\,\mathbf{v} = \mathbf{0}$ then

$A[\mathbf{v}]_{B_V} = [T\,\mathbf{v}]_{B_W} = [\mathbf{0}]_{B_W} = \mathbf{0}$.

This shows $[\mathbf{v}]_{B_V} \in \ker A$ & we've proved ($\Longrightarrow$).

Check ($\Longleftarrow$). Suppose$[\mathbf{v}]_{B_V} \in \ker A$. Then

$[T\,\mathbf{v}]_{B_W} = A[\mathbf{v}]_{B_V} = \mathbf{0}$.

i.e. $T\,\mathbf{v}$ has co-ords $\mathbf{0}$ so $T\,\mathbf{v} =$

$\therefore \mathbf{v} \in \ker T$ & a) is proved.

c) proved similarly.

b) & d) follow from a) & c) resp. & fact that bases can be checked via coords. Proof omitted.

e.g. 2 revisited Recall $T : \mathbb{P}_3(\mathbb{R}) \longrightarrow \mathbb{P}_2(\mathbb{R})$ represented wrt bases $B_3 = \{1, x, x^2, x^3\}, B_2 = \{1, x, x^2\}$ by

$$A = \begin{pmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

a) Find a basis for $\ker T$.

b) Find a basis for im T.

c) Verify the rank-nullity thm in this case directly.

A $\ker A$ consists of solns $\mathbf{x} \in \mathbb{R}^4$ to $A\,\mathbf{x} = \mathbf{0}$. A already in row echelon form so intro param corresp to non-leading columns $x_1 = \lambda, x_3 = \mu$. Gen soln is

$\mathbf{x} =$

so a basis for $\ker A$ is

Prop b) $\Longrightarrow$ there's a corresp basis for $\ker T$, namely

Hence $\operatorname{null} T =$

b) The 2nd & 4th col for A are the leading columns so

$\{[Tx]_{B_2}, [Tx^3]_{B_2}\}$ is a basis for im A (these are 2nd & 4th columns of A).

Prop d) $\implies$ we have a corresp basis for im T, namely

$\{Tx = -1, \;\; Tx^3 = 6x + 3x^2\}$

as can be seen by reading off the entries in 2nd & 4th column of A.

$\therefore$ rank $T = 2$.

c) $\mathrm{null}\, T + \text{rank } T = 2 + 2 = 4 = \dim \mathbb{P}_3$ so the rank-nullity thm is verified directly.