Lecture 16: Kernels & images

Aim Lecture Introduce kernels & images which inform us about the uniqueness & existence of solns to linear eqns.

Kernels

Defn Let $T : V \longrightarrow W$ be linear. The kernel of T is

$\ker T := \{\mathbf{v} \in V | T\,\mathbf{v} = \mathbf{0}\}$

i.e. "Homogeneous solns". See in prop 1 below that kernels are subspaces so can define the nullity of T to be $\text{null}\, T := \dim \ker T$.

If $A \in M_{mn}(\mathbb{F})$ then define the kernel of A, denoted $\ker A$ to be the kernel of the assoc linear map $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ i.e. solns to $A\,\mathbf{x} = \mathbf{0}$.

e.g. 1 $T = \frac{d}{dx} : \mathbb{P} \longrightarrow \mathbb{P}$ has kernel, polynomial solns to $\frac{dp}{dx} = 0$.

Gen soln is $p =$

$\ker T = \{c | c \in \mathbb{R}\} = \mathbb{R}$.

$\operatorname{null} T = \dim_{\mathbb{R}} \mathbb{R} =$

Kernels are subspaces

Propn 1 If $T : V \longrightarrow W$ is linear then $\ker T$ is a subspace of V.

Proof: We check closure axioms.

a) Prop 1 lect 14 shows $T\mathbf{0} = \mathbf{0}$

so $\mathbf{0} \in \ker T$.

b) If $\mathbf{v}, \mathbf{v}' \in \ker T$ then $\mathbf{0} = T\,\mathbf{v} = T\,\mathbf{v}'$. Check $\mathbf{v} + \mathbf{v}' \in \ker T$.

$T(\mathbf{v} + \mathbf{v}') =$

so $\ker T$ is closed under addn.

c) If $\lambda \in \mathbb{F}$ then

$T(\lambda\, \mathbf{v}) =$

so $\lambda\, \mathbf{v} \in \ker T$ & $\ker T$ is also closed under scalar multn.

Hence $\ker T \leq V$.

Dumb e.g. 2 Show

$W = \{\mathbf{x} \in \mathbb{R}^4\, | x_1 + 2x_2 + 3x_3 + 4x_4 = 0\}$

is a subspace of $\mathbb{R}^4$.

Ans: $W = \ker$

Prop 1 $\implies$ W is a subspace of $\mathbb{R}^4$.

Computing bases for $\ker A$

We've actually done this before!

e.g.3 Find a basis for $\ker A$ where

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 3 & 3 \end{pmatrix}.$$

A We solve $A\,\mathbf{x} = \mathbf{0}$.

$$A \longrightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ & & & \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ & & & \end{pmatrix}$$

Intro parameters for non-leading columns:

$\lambda = x_3, \mu = x_4$. Gen soln is

$\mathbf{x} =$

$=$

so a basis for $\ker A$ is

Inhomogeneous equations

The nature of solns to $T\,\mathbf{v} = \mathbf{w}$ is sim to $\mathbb{F}^m$ case as following prop shows.

Prop 2 Let $T : V \longrightarrow W$ be linear. Suppose given $\mathbf{w} \in W$ and a particular soln $\mathbf{v} = \mathbf{v}_p$ to eqn

$$(*) \qquad T\,\mathbf{v} = \mathbf{w}\,.$$

The complete set of solns to (*) is

$\mathbf{v} = \mathbf{v}_p + \mathbf{v}_h$, for any $\mathbf{v}_h \in \ker T$.

Proof: As in session 1. Don't believe me? Observe

$\mathbf{v}_h + \mathbf{v}_p$ is a soln since

Conversely, if $\mathbf{v}$ is a soln so $T\,\mathbf{v} = \mathbf{w}$

then $T(\mathbf{v} - \mathbf{v}_p) =$

$\implies \mathbf{v}_h := \mathbf{v} - \mathbf{v}_p \in \ker T$

& $\mathbf{v} = \mathbf{v}_p + \mathbf{v}_h$.

Cor If $\ker T = \mathbf{0}$ then T is 1-1.

Why? Let $\mathbf{w} \in W$. Prop 2 $\implies$ you can only vary a particular soln $\mathbf{v}_p$ to linear eqn $T\mathbf{v} = \mathbf{w}$ by some $\mathbf{v}_h \in \ker T$. So if $\ker T = \mathbf{0}$, the soln is unique (assuming it exists) i.e. T is 1-1.

Geometric example

E.g. 4 Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be orthogonal projn onto 1-dim subspace L.

$\ker T =$

Image

Defn-Prop Let $T : V \longrightarrow W$ be linear. The image of T,

$\text{im } T = T(V) := \{T\,\mathbf{v} \,|\, \mathbf{v} \in V\}$

$= \{\mathbf{w} \in W |\, \mathbf{w} = T\,\mathbf{v}, \text{for some } \mathbf{v} \in V\}.$

is a subspace of V.

We define rank $T := \dim \text{im } T$.

Proof: We check closure axioms.

a) $\text{im } T \ni T\mathbf{0} =$

b) For $\mathbf{w}, \mathbf{w}' \in \text{im } T$, there are $\mathbf{v}, \mathbf{v}'$

with $\mathbf{w} = T\,\mathbf{v}, \mathbf{w}' = T\,\mathbf{v}'$.

Then $\mathbf{w} + \mathbf{w}' =$

so im T is closed under addn.

c) If $\lambda \in \mathbb{F}$ & $\mathbf{w} = T\,\mathbf{v}$ then

$\lambda\,\mathbf{w} =$

so im T is closed under scalar multn too.

$\therefore$ im $T \leq W$.

Rem 1) im T consists of vectors $\mathbf{b}$ for which you can solve $T\mathbf{v} = \mathbf{b}$ with some $\mathbf{v} \in V$.

2) In calculus language, image is just the range of the function.

3) Proof above shows more gen that if $V_1 \leq V$ then $T(V_1) := \{T\mathbf{v} \,|\, \mathbf{v} \in V_1\}$ is a subspace of W.

e.g. 4 again $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ orthog projn onto 1-dim subspace L.

im $T =$

Computing bases for im A

Let $A \in M_{mn}(\mathbb{F})$ & $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ be the assoc lin map.

Define im $A =$ im T_A

$= \{T_A \, \mathbf{v} =$

$= \mathrm{col}(A).$

Define also rank $A =$ rank T_A

We've computed bases for im A before.

E.g.5

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$

$A \longrightarrow$

1st & 2nd columns leading so

a basis for im A is

& rank $A =$

Rank-Nullity theorem

Thm 1 (Matrix version)

Let $A \in M_{mn}(\mathbb{F})$ & U be a row echelon form for A. Then

a) $\operatorname{null} A =$ no. non-leading columns of U.

b) rank $A =$ no. leading columns of U.

c) (Rank-Nullity)

$n =$ no. col of $A = \operatorname{null} A +$ rank A.

Proof: a) Each non-leading column in U gives a param in gen soln to $A\mathbf{x} = \mathbf{0}$.

$\therefore$ gives a basis vector for $\ker A$.

$\therefore \operatorname{null} A =$ no. non-leading columns for A.

& a) holds.

b) Thm 1, lecture 12 gives basis

$\{\mathbf{v}_i = i\text{-th col of } A | i\text{-th col of } U \text{ is leading } \}$.

so b) holds.

c) Add a) & b).

Thm 2 Let $T : V \longrightarrow W$ be a linear map, V, W

finite dim.

Then $\dim V = \operatorname{null} T + \operatorname{rank} T$.

Proof omitted. Reduces to thm 1 using matrix reprn thm of next lecture.

e.g. 4 again A geometric picture illustrating rank-nullity thm

Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be an orthog projn onto 1-dim subspace L.

$\operatorname{null} T =$

$\operatorname{rank} T =$

$\dim \mathbb{R}^3 = 3.$

So $\dim \mathbb{R}^3 = \operatorname{null} T + \operatorname{rank} T$.

Applications of the rank-nullity theorem

Defn A hyperplane in $\mathbb{R}^n$ is the set of solns to an eqn of form

$a_1 x_1 + \ldots + a_n x_n = c.$

for some $a_i \in \mathbb{R}$ not all zero & $c \in \mathbb{R}$.

e.g. 6 We'll use the rank-nullity thm to show that the intersection of 2 hyperplanes through $\mathbf{0}$ in $\mathbb{R}^n$ has dim $\geq n-2$.

A Consider hyperplanes

$H_a : a_1x_1 + \ldots + a_nx_n = 0$

$H_b : b_1x_1 + \ldots + b_nx_n = 0.$

$H_a \cap H_b$ is the set of simultanous solns

i.e. this is kernel of

$A =$

$\therefore \dim H_a \cap H_b = \dim \ker A = \operatorname{null} A.$

But im $A \leq \mathbb{R}^2 \implies$ rank $A \leq \mathbb{R}^2 = 2.$

So $\operatorname{null} A = n -$ rank $A \geq n-2.$

ex Can you generalise this to m hyperplanes?