Lecture 15: Geometry & Algebra of linear maps.

Aim Lecture Study some geom examples of lin maps. See how the algebra of linear maps mimics the algebra of matrices.

Visualising some lin maps

e.g. 1 Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the lin map assoc to matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & .5 \end{pmatrix}$ i.e. $T(x,y)^T = (2x, .5y)^T$.

We can plot some values of T,

$T\,\mathbf{e}_1 =$ $\qquad\qquad$ $, T\,\mathbf{e}_2 =$

Looking at a unit square:

$\therefore$ T stretches out horizontally by a factor of 2 &

shrinks vertically to half.

Orthogonal projection

Let $\mathbf{u} \in \mathbb{R}^n$ be unit vector.

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be projn onto

line $L := \text{Span}(\mathbf{u})$.

Recall $T\,\mathbf{v} = \text{proj}_{\mathbf{u}}\,\mathbf{v} = (\mathbf{v} \,.\, \mathbf{u})\,\mathbf{u}$.

T is linear.

Proof 1: Add Condn: for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$

$T(\mathbf{v} + \mathbf{w}) = ((\mathbf{v} + \mathbf{w}) .\, \mathbf{u})\,\mathbf{u}$

$= (\mathbf{v} \,.\, \mathbf{u} + \mathbf{w} \,.\, \mathbf{u})\,\mathbf{u}$

$= (\mathbf{v} \,.\, \mathbf{u})\,\mathbf{u} + (\mathbf{w} \,.\, \mathbf{u})\,\mathbf{u}$

$= T\,\mathbf{v} + T\,\mathbf{w}$

so addn condn holds.

Scalar Multn: For $\lambda \in \mathbb{R}$,

$T(\lambda\, \mathbf{v}) =$

Hence scalar multn also holds & T is linear.

Proof 2: Geometric. e.g. see picture for addn condn.

Proof 3: $T\,\mathbf{v} = (\mathbf{u}^T\,\mathbf{v})\,\mathbf{u} = \mathbf{u}\,\mathbf{u}^T\,\mathbf{v}$

(since $\mathbf{u}^T\,\mathbf{v}$ is a scalar so commutes with $\mathbf{u}$).

$\therefore T$ is the lin map assoc to the matrix $A =$

Linear combns of lin maps

Defn 1 Let $T, T' : V \longrightarrow W$ be lin maps.

Let $\lambda, \mu \in \mathbb{F}$. Define new maps

Sum: $T + T' : V \longrightarrow W$ by

$$(T + T')\,\mathbf{v} = T\,\mathbf{v} + T'\,\mathbf{v}.$$

scalar multiple: $\lambda T : V \longrightarrow W$ by

$(\lambda T)\,\mathbf{v} = \lambda(T\,\mathbf{v})$.

Linear Combn: $\lambda T + \mu T' : V \longrightarrow W$ by

$(\lambda T + \mu T')\,\mathbf{v} = \lambda(T\,\mathbf{v}) + \mu(T'\,\mathbf{v})$.

Propn 1 $T + T', \lambda T$ are linear too. Hence so is $\lambda T + \mu T'$.

Proof: Check λT lin: for $\mathbf{v}, \mathbf{w} \in V, \alpha \in \mathbb{F}$

Add Condn: $(\lambda T)(\mathbf{v} + \mathbf{w}) \overset{\text{def}}{=}$

$= \lambda(T\,\mathbf{v} +$

$= \lambda(T\,\mathbf{v})$

$=$

so addn condn holds.

Scalar Multn Condn: $(\lambda T)(\alpha\,\mathbf{v})$

$= \lambda(T(\alpha\,\mathbf{v})) = \lambda(\alpha T\,\mathbf{v})$

$= \alpha\lambda(T\,\mathbf{v}) = \alpha((\lambda T)\,\mathbf{v})$

so scalar multn condn also holds & T is linear.

Check $T+T'$ linear: Can prove as above. Here we only prove in case $V=\mathbb{F}^n, W=\mathbb{F}^m$ so by matrix reprn thm there are $A,B\in M_{mn}(\mathbb{F})$ such that for any $\mathbf{v}\in V$ we have

$T\,\mathbf{v}=A\,\mathbf{v},\quad T'\,\mathbf{v}=B\,\mathbf{v}.$

$(T+T')\,\mathbf{v}=T\,\mathbf{v}+T'\,\mathbf{v}=A\,\mathbf{v}+B\,\mathbf{v}=(A+B)\,\mathbf{v}.$

$T+T'$ is lin map assoc to the matrix $A+B$ so is linear.

Proof gives

Formula 1 $T_A+T_B=T_{A+B}$

Sim $\lambda T_A+\mu T_B=T_{\lambda A+\mu B}$.

where T_A, T_B etc are lin maps assoc with matrices A,B etc.

Application to orthogonal projections

e.g. 2 Identity linear map

For vect space V, let $\mathrm{id} : V \longrightarrow V$ be the function $\mathrm{id}\,\mathbf{v} = \mathbf{v}$. It is linear (CHECK!).

$\mathrm{id} : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ is represented by

since $\mathrm{id}\,\mathbf{v} = \mathbf{v} =$

e.g. 3 Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ be an o/n basis.

Let $P_i :=$ projn onto line $\mathrm{Span}(\mathbf{v}_i)$.

What's $P_{12} :=$ projn onto plane $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$?

Method 1: $P_{12}\,\mathbf{v} = \mathbf{v} - \mathrm{proj}_{\mathbf{v}_3}\,\mathbf{v}$

$\mathrm{id}\,\mathbf{v} - P_3\,\mathbf{v} = (\mathrm{id} - P_3)\,\mathbf{v}$.

$\therefore P_{12} = \mathrm{id} - P_3$

so is linear being a lin combn of the lin maps id &

P_3.

Method 2: also $P_{12}\,\mathbf{v} = P_1\,\mathbf{v} + P_2\,\mathbf{v}$

Above show P_{12} is matrix multn by

$I_3 - \mathbf{v}_3\,\mathbf{v}_3^T =$

Composing & inverting linear maps

Propn 2 Let $T : V \longrightarrow W, S : U \longrightarrow V$ be linear maps. Then $T \circ S : U \longrightarrow W$ is linear.

(Recall $(T \circ S)\,\mathbf{u} = T(S\,\mathbf{u})$.)

Proof: ex in checking axioms, though following formula should convince you it's true.

Formula 2 For $A \in M_{lm}(\mathbb{F}), B \in M_{mn}(\mathbb{F})$ we have $T_A \circ T_B = T_{AB}$.

Why? For any $\mathbf{u} \in \mathbb{F}^n$ we have

$(T_A \circ T_B)\,\mathbf{u} = T_A(T_B\,\mathbf{u}) = T_A(B\,\mathbf{u}) = AB\,\mathbf{u} =$

$T_{AB}\,\mathbf{u}$.

Propn 3 If $T : V \longrightarrow W$ is linear and invertible then

its inverse $T^{-1} : W \longrightarrow V$ is also linear.

Proof: ex in checking axioms though following formula should convince you it's true.

Formula 3 For an invertible matrix A,

$(T_A)^{-1} = T_{A^{-1}}$.

Why? Given vectors $\mathbf{x}, \mathbf{y}$ with $T_A\,\mathbf{x} = \mathbf{y}$, we see $A\,\mathbf{x} = \mathbf{y}$.

$\therefore \mathbf{x} = A^{-1}\,\mathbf{y} = T_{A^{-1}}\,\mathbf{y}$.

This shows $(T_A)^{-1} = T_{A^{-1}}$.

Example of rotations

Let $T_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be anti-clockwise rotation about $(0, 0)$ through angle θ.

What's $T\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$?

On can check geom that T_θ is linear e.g. scalar multn condn $T(\lambda\,\mathbf{v}) = \lambda T\,\mathbf{v}$ for $\lambda \in \mathbb{R}, \mathbf{v} \in \mathbb{R}^2$ follows from picture.

Matrix reprn thm $\implies$ T_θ must be the lin map assoc to the matrix

$R_\theta := (T\,\mathbf{e}_1\,T\,\mathbf{e}_2)$.

From picture

see $T\,\mathbf{e}_1 =$

so T is the lin map assoc to the matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

e.g. 4 Prop & formula 3 can be easily verified directly for the rotation T_θ.

Geometrically we see T_θ is invertible with "inverse" rotation $T_\theta^{-1} =$

Hence the inverse is linear too (being another rotn).

We now check the matrix representing T_θ^{-1} is indeed R_θ^{-1}.

$$R_\theta^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

which is indeed the matrix representing