Lecture 14: Linear maps or transformations

Recall, a (homogeneous) linear fn $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is one of the form $f(x_1, \ldots, x_n)^T = a_1x_1 + \ldots + a_nx_n$ for some $a_1, \ldots, a_n \in \mathbb{R}$.

Aim Lecture Generalise this notion to linear maps between arbitrary vector spaces.

Definition of linear map

Defn Let V, W = vector space / $\mathbb{F}$ and $T : V \longrightarrow W$ a function. We say T is a linear map or transformation if the following 2 condns hold:

a. (Addition Condn) For any $\mathbf{v}, \mathbf{w} \in V$

$T(\mathbf{v} + \mathbf{w}) = T\,\mathbf{v} + T\,\mathbf{w}$

i.e. T preserves

& b. (Scalar Multn Condn) For any $\lambda \in \mathbb{F}$

$T(\lambda\,\mathbf{v}) = \lambda(T\,\mathbf{v})$

i.e. T preserves

Examples

e.g. 1 We show that the fn $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x_1, x_2)^T = (2x_1 + 3x_2, x_1)^T$ is linear.

How? Below let $(x_1, x_2)^T, (y_1, y_2)^T \in \mathbb{R}^2$, $\&\lambda \in \mathbb{R}$ be arbitrary.

Check addn condn: $T((x_1, x_2)^T + (y_1, y_2)^T)$

$$= T(x_1 + y_1, x_2 + y_2)^T$$

$$= \begin{pmatrix} 2(x_1 + y_1) + 3(x_2 + y_2) \\ x_1 + y_1 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 \end{pmatrix} + \begin{pmatrix} 2y_1 + 3y_2 \\ y_1 \end{pmatrix}$$

$$= T(x_1, x_2)^T + T(y_1, y_2)^T$$

so the addn condn holds.

Check scalar multn condn: $T(\lambda(x_1, x_2)^T)$

$$= T(\lambda x_1, \lambda x_2)^T$$

$$= \begin{pmatrix} 2\lambda x_1 + 3\lambda x_2 \\ \lambda x_1 \end{pmatrix}$$

$$= \lambda(2x_1 + 3x_2, x_1)^T = \lambda T(x_1, x_2)^T$$

so the scalar multn condn holds too.

Since both the addn condn & scalar multn condn hold, T is linear.

non- e.g. 2

$T : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $T(x) = x^2$ is not linear.

Why? If it were lin then scalar multn condn $\Longrightarrow$

$T(2 \times 1) = 2T(1)$.

BUT $T(2) = 2^2 = 4$ while $2T(1) =$

Since $T(2) \neq 2T(1)$, the scalar multn condn fails & T is not linear.

e.g. 3 C = vector space of continuous $\mathbb{R}$-valued fns on $\mathbb{R}$.

C^1 = subspace of continuously differentiable fns.

Define $T : C^1 \longrightarrow C$ to be differentiation i.e.

$Tf = f'$.

T is linear.

Why? Addn: for $f, g \in C^1$

$T(f + g) =$

so addn condn holds

Scalar Multn: for $\lambda \in \mathbb{R}$

$T(\lambda f) =$

Hence, scalar multn condn also

Linear maps assoc to matrices

stereotypical e.g. 4 Let $A \in M_{mn}(\mathbb{F})$. Define $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ by $T_A(\mathbf{v}) = A\,\mathbf{v}$.

This is linear & is called the lin map assoc to A.

Proof: (Addn condn) for $\mathbf{v}, \mathbf{w} \in \mathbb{F}^n$

$T_A(\mathbf{v} + \mathbf{w}) =$

so addn condn holds

(Scalar Multn Condn) if also $\lambda \in \mathbb{F}$

$T_A(\lambda\, \mathbf{v}) =$

giving scalar

Since the addn & scalar multn condn

sub-e.g. If $A = (a_1\ a_2\ \ldots$

Basic properties of linear maps

Propn 1 Let $T : V \longrightarrow W$ be linear. Then

a. $T\mathbf{0} = \mathbf{0}$ b. $T(-\,\mathbf{v}) = -T\,\mathbf{v}$.

Proof:

a.

b.

Warning Part a) $\implies$ a lin fn $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = mx + b$

is not a lin map of vect spaces unless

Prop 2 (Preservation of Lin Combn) Let $T : V \longrightarrow W$ be linear. Then for

scalars $\lambda_1, \ldots, \lambda_n$,

vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$

we have

$T(\lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n) = \lambda_1 T \mathbf{v}_1 + \ldots + \lambda_n T \mathbf{v}_n$

Proof: By induction on n. Just do $n = 2$ case here

$T(\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2) = T(\lambda_1 \mathbf{v}_1) +$

$=$

Matrix representation thm

Remark Prop 2 $\implies$ if you know $T : V \longrightarrow W$ is linear and the values of T on a spanning set for V then T is determined.

To illustrate this,

E.g. 5 Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ linear &

$T(1,1)^T = (1,1,1)^T, T(1,0)^T = (2,2,1)^T$.

Note $(1,1)^T, (1,0)^T$ span $\mathbb{R}^2$.

Then $T(0,1)^T = T(\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix})$

$$= T\begin{pmatrix} 1 \\ 1 \end{pmatrix} - T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}.$$

$T(\lambda, \mu)^T = T(\lambda(1,0)^T + \mu(0,1)^T)$

$=$

$=$

$= A \begin{pmatrix} \lambda \\ \mu \end{pmatrix}$

where $A =$

e.g. generalises to

Thm 2 (Matrix Representation Thm)

Let $T : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ be linear.

$\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{F}^n$ be the standard basis vectors.

Define $\mathbf{a}_1 = T\,\mathbf{e}_1, \ldots, \mathbf{a}_n = T\,\mathbf{e}_n$.

& $A = (\mathbf{a}_1\, \mathbf{a}_2 \ldots \mathbf{a}_n)$.

Then $T = T_A$, the lin map assoc to the matrix A

i.e. for all $\mathbf{x} \in \mathbb{F}^n, T\,\mathbf{x} = A\,\mathbf{x}$.

Proof: $T(x_1, \ldots, x_n)^T$

$= T(x_1\,\mathbf{e}_1 +$

$= x_1 T\,\mathbf{e}_1 +$

$= x_1$

e.g. 6 revisited