Lecture 13: Data fitting. Review functions.

Aim Lecture We observe role of linear algebra in data fitting. We review invertible fns.

Motivational example

e.g. 1 Lex believes

Experiment:

r	1	2	3	4
Strength	9.7	9.2	7.9	4.9

Typical questions:

Interpolation: estimate strength when r is between data values, e.g. $r = 3.5$

Extrapolation: estimate strength when r is outside range of data, e.g. $r = 5$.

Objective Find a cubic fn

$y(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3$ such that

$y(1) = 9.7, y(2) =$

Point These 4 eqns give 4 lin eqns in a_0, a_1, a_2, a_3 with which to determine poly $y(r)$. Can then use $y(r)$ to estimate strength for other values of r other than $1, 2, 3, 4$.

Polynomial interpolation

Consider data points $(t_0, y_0), \ldots, (t_n, y_n) \in \mathbb{R}^2$ where $t_0, \ldots, t_n$ are distinct.

Seek polynomial

$y(t) = \lambda_0 + \lambda_1 t + \ldots + \lambda_n t^n$

s.t. $y_i = y(t_i)$

Notn: $\mathbf{t} = (t_0, t_1, \ldots, t_n)^T$

$y(\mathbf{t}) = (y(t_0), y(t_1), \ldots, y(t_n))^T$

$\mathbf{y} =$

$\lambda =$

Want to find fn $y(t)$ with $y(\mathbf{t}) = \mathbf{y}$. If

$$A := \begin{pmatrix} 1 & t_0 & t_0^2 & \dots & t_0^n \\ 1 & t_1 & t_1^2 & \dots & t_1^n \\ \vdots \\ 1 & t_n & t_n^2 & \dots & t_n^n \end{pmatrix}$$

then $A\lambda =$

$= y(\mathbf{t})$

Upshot The polynomials $y(t)$ which "fit the data", i.e. with $y(\mathbf{t}) = \mathbf{y}$

are those whose coeff $\lambda_0, \lambda_1, \dots, \lambda_n$ solve $A\lambda = \mathbf{y}$.

Thm 1 A is invertible so there's a unique poly

$y(t) = \lambda_0 + \lambda_1 t + \ldots + \lambda_n t^n$

in $\mathbb{P}_n$ s.t. $y(t_i) = y_i$. It's coeff are given $\lambda =$

Proof: Cor a) lect 4 $\implies$ at most 1 poly of degree

$\leq n$ can fit the data.

$\therefore$ if $A\lambda = \mathbf{y}$ has solns then it is unique.

$\therefore$ columns of

$\therefore$ row-echelon form has all

But A is square so all rows are leading too and A must be invertible.

N.B. $\dim \mathbb{P}_n =$ no. data points.

e.g. 1 cont'd see MATLAB lexl.m

Data points: $(t_i, y_i) =$

$(1, 9.7), (2, 9.2), (3, 7.9), (4, 4.9)$

$\mathbf{t} = (1, 2, 3, 4)^T, \mathbf{y} =$

$A =$

The coeff of the cubic fn $y(t)$ which fits the data is

$\lambda = A^{-1}\mathbf{y} =$

i.e. $y(t) =$

Lagrange polynomials

Previously found $y(t)$ by finding

its coeff = coords wrt basis

Here construct more natural basis.

Consider $\mathbf{t} = (t_0, \ldots, t_n)^T$ with t_i distinct.

For $j = 0, \ldots, n$ consider Lagrange polynomials

$$P_j(t) := \prod_{k \neq j} \frac{t - t_k}{t_j - t_k} \in \mathbb{P}_n$$

This is a product of n linear factors.

e.g. 2 If $\mathbf{t} = (1, 2, 3)$ then

$P_0(t) = \frac{(t - t_1)}{(t_0 - t_1)} \frac{(t - t_2)}{(t_0 - t_2)}$

$P_1(t) =$

$P_2(t) =$

Thm 2 a) $P_j(t_j) =$

b) $P_j(t_i) =$

c) $B = \{P_0(t), \ldots, P_n(t)\}$ is a basis for $\mathbb{P}_n$.

d) If $y(t) \in \mathbb{P}_n$ fits the data i.e. $y(\mathbf{t}) = \mathbf{y}$ then

$(*) \quad y(t) = y_0 P_0(t) + \ldots + y_n P_n(t)$.

i.e. $[y(t)]_B = (y_0, \ldots, y_n)^T = \mathbf{y}$.

Proof: a) & b) follow on substn.

c) $\dim \mathbb{P}_n = n + 1$. $B \subset \mathbb{P}_n$ also has $n + 1$ vectors so suffice show it is lin indep.

Suppose $\lambda_0 P_0(t) + \ldots + \lambda_n P_n(t) = 0$.

For any $i = 0, \ldots, n$,

$0 = \lambda_0 P_0(t_i) + \ldots + \lambda_i P_i(t_i) + \ldots + \lambda_n P_n(t_i)$

$=$

$\therefore P_0(t), \ldots, P_n(t)$ are lin indep. $\therefore B$ is a basis.

d) Just note

$y_0 P_0(t_i) +$

so both sides of (*) have the same values for the

$n+1$ inputs $t_0, \ldots, t_n$. But both sides are also polys of degree $\leq n$ so, since they agree for $n+1$ different values, they must be the same.

N.B. Thm 2d) means don't need to solve eqns to find $y(t)$. BUT you need to work to compute $P_0(t), \ldots, P_n(t)$.

e.g. 1 again If $\mathbf{t} = (1, 2, 3, 4)^T$ then desired cubic fn can also be written as

$y(t) = 9.7P$

$P_0(t), P_1(t), P_2(t), P_3(t)$ are the appropriate Lagrange polys.

Interpolation by general functions

Gen setup. Consider data points

$(t_1, y_1), \ldots, (t_n, y_n)$

Let $\phi_1, \ldots, \phi_n \in \mathcal{R}[\mathbb{R}]$ be lin indep.

Q Find $y(t) \in \mathrm{Span}(\phi_1, \ldots, \phi_n)$

s.t. $y(t_i) = y_i$ for all i i.e. $y(\mathbf{t}) = \mathbf{y}$.

A As in poly case, write

$y(t) = \lambda_1\phi_1 + \ldots + \lambda_n\phi_n$.

$A =$

Then $y(\mathbf{t}) = A\lambda$ so solving for $y(t)$ amounts to solving $A\lambda = \mathbf{y}$.

e.g.2 Market with price trend anticipation.

Price $p(t)$ at time t governed by a 2nd order ODE like

$$\frac{d^2p}{dt^2} - 3\frac{dp}{dt} + 2p = 0$$

Find $p(t)$ if $p(0) = 7, p(1) = 8$

A Char eqn $\lambda^2 - 3\lambda + 2 = 0$

$\implies p(t) = \lambda_1 e^t + \lambda_2 e^{2t} \in \text{Span}(e^t, e^{2t})$

Now just solve for λ_1, λ_2 using cont'd

Invertible functions

Recall following defns from calculus regarding a function $f : X \longrightarrow Y$.

Defn 1) We say that f is one-to-one (1-1) or injective if for any $y \in Y$, the soln to $f(x) = y$ is

i.e. $f(x) = f(x') \implies$

2) $f : X \longrightarrow Y$ is onto or surjective if for any $y \in Y$, a soln to $f(x) = y$

i.e. $\text{im} f$

Recall also from calculus

Facts a) A fn $f : X \longrightarrow Y$ is invertible iff f is 1-1 &

b) In this case, the eqn $f(x) = y$ always has a denoted $x =$

c) $f \circ f^{-1} =$

e.g. 3 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $f(x_1, x_2) = (x_2, 2x_1)$ is invertible with inverse $f(y_1, y_2) =$