Lecture 12: Constructing Bases

Aim Lecture Give effective means of finding basis.

Bases for Span$(S) \subseteq \mathbb{F}^m$

Saw one can reduce spanning set to a basis in theory.

Next thm shows how to do this in practice.

Thm 1 Let $A = (\mathbf{v}_1\, \mathbf{v}_2 \ldots \mathbf{v}_n)$. Recall

col$(A) =$

Let U be a row echelon form for A.

Then we have the following basis of col(A)

$B = \{\mathbf{v}_i \,|i -$ th column of U is leading$\}$.

Proof: will be clear from following e.g. (also see notes §7.6.3 thm 6)

e.g. 1 Find a basis for col(A) where

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & -3 & 2 & 1 \end{pmatrix} = (\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4)$$

Ans

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & -3 & 2 & 1 \end{pmatrix} \longrightarrow$$

$$\longrightarrow$$

1st & 2nd columns are leading.

Thm 1 $\implies$ $\{\mathbf{v}_1 = (1,1,1)^T, \mathbf{v}_2 = (-1,1,-2)^T\}$ is a basis for $\mathrm{col}(A) = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.

Why did it work?

i.e. why's $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathrm{col}(A)$?

Check lin indep: This is clear since they are not parallel. We will however check from defns so you can see why the method given in thm 1 works. We need to show the only soln to $x_1\,\mathbf{v}_1 + x_2\,\mathbf{v}_2 = \mathbf{0}$ is $x_1 = x_2 = 0$.

We solve $x_1\,\mathbf{v}_1 + x_2\,\mathbf{v}_2 = 0$ by just omitting 3rd & 4th column from above calculation,

$\therefore$ only soln to is $x_1 = x_2 = 0$

& $\mathbf{v}_1, \mathbf{v}_2$ are lin indep.

N.B. Argument shows why vectors corresp to leading columns of U in thm are lin indep.

Check span: 3rd & 4th columns correspond to parameters in soln to $x_1\,\mathbf{v}_1 + x_2\,\mathbf{v}_2 + x_3\,\mathbf{v}_3 + x_4\,\mathbf{v}_4 = \mathbf{0}$.

Pick soln $\mathbf{x}$ with $x_3 = 1, x_4 = 0$.

Back substn $\implies$ $\mathbf{x} = (-1/2, 1/2, 1, 0)^T$.

$\mathbf{0} = A\,\mathbf{x} = -\frac{1}{2}\,\mathbf{v}_1 + \frac{1}{2}\,\mathbf{v}_2 + \mathbf{v}_3$

$\therefore \mathbf{v}_3 = \frac{1}{2}\,\mathbf{v}_1 - \frac{1}{2}\,\mathbf{v}_2 \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$

so we can delete $\mathbf{v}_3$ from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ without shrinking the span.

Sim, setting $x_3 = 0, x_4 = 1$

get soln $\mathbf{x} =$

so $\mathbf{0} = A\,\mathbf{x} =$

& $\mathbf{v}_4 \in$

so we can also delete $\mathbf{v}_4$ without shrinking the span.

i.e. $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$

so $\mathbf{v}_1, \mathbf{v}_2$ span.

Hence, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is lin indep & spans $\mathrm{col}(A)$ so is a basis.

Extending linearly independent sets in $\mathbb{F}^m$

Thm 2 Let W be a subspace of $\mathbb{F}^m$ and $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset W$ be lin indep.

Suppose $\{\mathbf{w}_1, \ldots, \mathbf{w}_r\}$ is a basis for W.

Then applying method of thm 1 to

$\{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}_1, \ldots, \mathbf{w}_r\}$

yields a basis for W which contains S i.e. extends S to a basis.

Note: 1) We can apply thm 1 since

$\{\mathbf{v}_1, \ldots, \mathbf{w}_r\}$ span W (in fact the last r vectors already span).

2) $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ lin indep $\implies$ row echelon form for $A = (\mathbf{v}_1 \ldots \mathbf{v}_n \, \mathbf{w}_1 \ldots \mathbf{w}_r)$ has 1st n columns leading.

$\therefore$ the basis B produced by this method

Hopefully, the reason why this works will be clear from the following e.g.

e.g. 2 Let $S = \{\mathbf{v}_1 = (1,2,-1)^T, \mathbf{v}_2 = (3,2,-1)^T\}$. Extend S to a basis of $\mathbb{R}^3$.

Ans: Note vectors not parallel $\implies$ S lin indep.

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Let $A = (\mathbf{v}_1\, \mathbf{v}_2\, \mathbf{e}_1\, \mathbf{e}_2\, \mathbf{e}_3)$

$$= \begin{pmatrix} 1 & 3 & 1 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{pmatrix} \longrightarrow$$

$\longrightarrow$

1st, 2nd & 4th columns are leading so thm 1 $\implies$

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_2\}$ is a basis for $\mathbb{R}^3$ containing $\{\mathbf{v}_1, \mathbf{v}_2\}$.

N.B. $\dim \mathbb{R}^3 = 3$ so we need to add one vector to get basis.

Remark What happens to this method if S is lin dependent so that S cannot form part of a basis?

Ans: Method produces a basis with as many members of S as possible.

Bases for subspaces defined by equations

e.g. 3 Let

$$A = \begin{pmatrix} 1 & 0 & 3 & 1 & -1 \\ 0 & 1 & 2 & -1 & 4 \end{pmatrix}.$$

You can check closure axioms to see $W := \{\mathbf{x} \in \mathbb{R}^5 \,|\, A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^5$.

What's a basis?

A General soln to $A\mathbf{x} = \mathbf{0}$ is

$\mathbf{x} =$

$=$

Thus a basis for W is $B =$

Why? Certainly every $\mathbf{x} \in W$ is a lin combn of B so B spans.

In fact, such an $\mathbf{x}$ is a unique lin combn of B

since the scalars λ, μ, ν are uniquely determined by x_3, x_4, x_5.

$\therefore B$ is a basis for W.

Subspaces of $M_{mn}(\mathbb{F})$ and $\mathbb{P}_d$

We reduce to the $\mathbb{F}^n$ case via coordinates.

Thm 3 Let B be a basis for a vect space V & $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$.

Let $A = ([\mathbf{v}_1]_B [\mathbf{v}_2]_B \ldots [\mathbf{v}_n]_B)$.

Let U be a row echelon form for A. Then we have the following basis of $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$

$B = \{\mathbf{v}_i \,|i-\text{th column of } U \text{ is leading}\}$.

Why? (Not exam). Linearity of coords $\implies$ lin relns in the $\mathbf{v}_i$ correspond to lin relns in the $[\mathbf{v}_i]_B$. Thus can repeat the argument for why thm 1 holds true.

e.g. 5 Find basis for $W = \text{Span}(p_1(x) = 1 + x +$

$x^2, p_2(x) = -1 + x - 3x^2, p_3(x) = 1 + 2x^2, p_4(x) = 2 + 3x + x^2)$.

A Note the polys all lie in $\mathbb{P}_2$ so we work in this vector space. Use basis $B = \{1, x, x^2\}$ for $\mathbb{P}_2$.

$$A = ([p_1(x)]_B [p_2(x)]_B [p_3(x)]_B [p_4(x)]_B)$$

$$= \begin{pmatrix} 1 & & 1 & 2 \\ 1 & 1 & & 3 \\ 1 & -3 & 2 & \end{pmatrix} \longrightarrow$$

as in e.g. 1.

First 2 columns are leading so a basis for W is