Lecture 11: Dimension

Aim Lecture A basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V gives n-dim coord system. Suggests V is n-dimensional. Need theory to ensure any two bases have the same number of vectors.

How does Span(S) vary with S

Prop V = vector space/ field $\mathbb{F}$

Let $S_1 \subseteq S_2$ be finite subsets of V. Then

$\text{Span}(S_1) \subseteq \text{Span}(S_2)$.

Proof: We may suppose $S_1 = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}, S_2 = \{\mathbf{v}_1, \ldots, \mathbf{v}_m, \ldots, \mathbf{v}_n\}$.

Need show every lin combn of S_1 is also a lin combn of S_2. Consider a lin combn of S_1

$\lambda_1 \mathbf{v}_1 + \ldots + \lambda_m \mathbf{v}_m =$

$\lambda_1 \mathbf{v}_1 + \ldots + \lambda_m \mathbf{v}_m + 0\, \mathbf{v}_{m+1} + \ldots + 0\, \mathbf{v}_n$

which is also a lin combn of S_2. Hence every vector in $\mathrm{Span}(S_1)$ is also in $\mathrm{Span}(S_2)$ and the prop is proved.

Thm 1 Let $\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ = vector space/ field $\mathbb{F}$. Then

$$(*) \qquad \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}) = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$$

iff $\mathbf{v} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

Proof: ($\Longrightarrow$) If (*) holds then

$\mathbf{v} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}) = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

($\Longleftarrow$) Suppose $\mathbf{v} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ so say

$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n$

for some scalars $\alpha_1, \ldots, \alpha_n$. We first show

$\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}) \subseteq \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. An arbitrary elt of

$\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v})$ has form

$\lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n + \lambda \mathbf{v} =$

$= (\lambda_1 + \alpha_1 \lambda)\, \mathbf{v}_1 + \ldots + (\lambda_n + \alpha_n \lambda)\, \mathbf{v}_n$

which lies in $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

Hence, $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}) \subseteq \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$

Prop 4 $\implies$

$\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}) \supseteq \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$

The two inclusions show $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}) = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

Shrinking spanning sets to bases

Rem If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ lin depend then some $\mathbf{v}_i \in \mathrm{Span}(S - \{\mathbf{v}_i\})$ (by alt charn of lin depend). Prop $\implies$

$\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) =$

$\mathrm{Span}(\mathbf{v}_1, \ldots \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n)$

i.e. can omit $\mathbf{v}_i$ without from S without shrinking its span.

e.g. 1 Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ all lie in plane

$P \subseteq \mathbb{R}^3$ & no two are parallel.

Span$(S) = P$. Above remark shows that you can remove any of the 3 vectors from S and the remaining two will still span P. This is clear geometrically from picture.

Thm 2 Suppose $S \subset V$ is a finite spanning set for V. Then some subset of S is a basis for S.

Why? We run following algorithm which keeps deleting vectors from S until you arrive at a basis.

Step 1: Ask if some $\mathbf{v} \in S$ is a lin combn of other vectors in S.

Step 2: If no, then S is lin indep by alt charn of lin depend. Done as S is lin indep spanning set & $\therefore$ a basis.

Step 3: If yes, delete $\mathbf{v}$ from S & note by remark that we still have $\text{Span}(S) = V$. Go back to step 1.

Note, process stops $\because$ S has only finitely many vectors and you can't keep deleting forever.

Dimension

Lemma Let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{F}^n$ be lin indep. Then $m \leq n$.

Proof: Let $A = (\mathbf{v}_1 \ldots \mathbf{v}_m) \in M_{nm}(\mathbb{F})$.

$\mathbf{v}_1, \ldots, \mathbf{v}_m$ lin indep

$\implies x_1 \mathbf{v}_1 + \ldots + x_m \mathbf{v}_m = \mathbf{0}$

has unique soln $0 = x_1 = \ldots = x_m$.

i.e. $A\mathbf{x} = \mathbf{0}$ has unique soln $\mathbf{x} = \mathbf{0}$.

$\therefore$ no. rows of $A \geq$ no. columns of A.

(Otherwise row echelon form for A has non-leading column so gen soln for $\mathbf{x}$ has a parameter in it, a contradiction.)

$\therefore n \geq m$.

Thm 3 Let V = vector space/ field $\mathbb{F}$

Let $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ span V.

Let $I = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subseteq V$ be lin indep.

Then $m \leq n$.

Proof: By thm 2, we can shrink S to basis B with $d \leq n$ vectors.

Scholium in lecture 10 $\implies$

$[\mathbf{v}_1]_B, \ldots, [\mathbf{v}_m]_B \in \mathbb{F}^d$ are lin indep.

Lemma $\implies$ $m \leq d$ so $m \leq n$ too.

Prop - Defn Let V = vector space/ field $\mathbb{F}$

Suppose $B_1 = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ &

$B_2 = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ are bases for V. Then $m = n$

In this case, say V is finite dimensional

& has dimension $\dim_{\mathbb{F}} V = \dim V := n$.

Proof: By thm 3

B_1 spans V & B_2 lin indep $\Longrightarrow$

B_2 spans V & B_1 lin indep $\Longrightarrow$

so $m = n$.

e.g. 2 For $\mathbf{v} \in \mathbb{R}^3 - \mathbf{0}$, the line

$V = \mathrm{Span}(\mathbf{v})$ has basis $B = \{\mathbf{v}\}$.

$\therefore \dim_{\mathbb{R}} V =$ no. elts of $B = 1$.

So a line is 1-dimensional as one would want.

e.g. 3 We have standard basis

$\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ so $\dim \mathbb{F}^n = n$.

e.g. 4 $\mathbb{C}$ is a vect space / $\mathbb{R}$ with

vector addn = addn of complex numbers

scalar multn = product of complex number by real.

$\mathbb{C}$ has basis (over $\mathbb{R}$)

since every complex number can be written uniquely as $a + bi$ for some (unique choice of) $a, b \in \mathbb{R}$.

Hence $\dim_{\mathbb{R}} \mathbb{C} =$

Note $\dim_{\mathbb{C}} \mathbb{C} = \dim_{\mathbb{C}} \mathbb{C}^1 = 1$.

E.g. 5 $\dim M_{mn}(\mathbb{F}) = mn$ since

$\{E_{11}, \ldots, E_{1m}, E_{21}, \ldots, E_{m1}, \ldots, E_{mn}\}$ is a basis.

$\dim \mathbb{P}_n$

Theoretical implications of dimension

Cor 1 For $V =$ vector space/ field $\mathbb{F}$ of dim d,

a) Any lin indep set I has $\leq d$ elts.

b) Any spanning set S has $\geq d$ elts.

Proof: a) V has a basis S with d elts. S is also a spanning set so

thm 3 $\implies d \geq$ no. elts of I.

b) Sim.

e.g. 5 $\dim \mathbb{R}^3 = 3$ so can't span $\mathbb{R}^3$ with < 3 vectors and any 4 vectors in $\mathbb{R}^3$ are lin dependent.

Cor 2 Let V = vector space/ field $\mathbb{F}$. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ ($\mathbf{v}_i$'s distinct). The following conditions on B are equivalent:

a) B is a basis for V.

b) B is lin indep & $n = \dim V$.

c) B spans V & $n = \dim V$.

Proof: Omitted, see notes §7.6, thm 3.

e.g. 6 Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an o/n set of vectors in $\mathbb{R}^n$.

We saw lect 10 that B is lin indep so cor 2 $\implies B$ is in fact a basis.

Existence of basis

Thm 4 Let V = vector space/ field $\mathbb{F}$

Let $S \subset V$ be a spanning set of n elts. Any subspace W of V has a basis with $\leq n$ vectors.

If $\dim W = \dim V$ then $W = V$.

Proof: Omitted. It's dual to proof thm 2.

e.g. 7 One can check closure axioms to see

$W := \{\mathbf{x} \in \mathbb{R}^5 \,|\, x_1 + 2x_2 - x_5 = 0\}$

is a subspace of $\mathbb{R}^5$.

Thm 4 $\implies$ it has a basis with $\leq \dim \mathbb{R}^5 = 5$ vectors. In fact, since $W \neq \mathbb{R}^5$, any basis has < 5 vectors.