Lecture 10: Bases & Coordinates

Aim Lecture Show how to set up coord systems using bases.

Basis

Defn 1 Let V = vect space / field $\mathbb{F}$. $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is a basis for V if

1) $\text{Span}(B) = V$

& 2) B is lin indep.

This is equiv to

(*) any $\mathbf{v} \in V$ can be written uniquely as a lin combn of B.

Why? Thm lect 9 says, if B is a basis, then any $\mathbf{v} \in \text{Span}(B) = V$ can be written uniquely as a lin combn of B.

Conversely,(*) $\implies$ $\text{Span}(B) = V$

& the unique way of writing $\mathbf{0}$ as a lin combn of B is

$\mathbf{0} = 0\,\mathbf{v}_1 + \ldots + 0\,\mathbf{v}_n$

i.e. B is also lin indep. $\therefore$ B is a basis.

e.g. 1 $V = \mathbf{0}$ has basis $\varnothing$.

e.g. 2 $\mathbb{P}_n$ has basis $B = \{1, x, x^2, \ldots, x^n\}$

Why? B spans $\mathbb{P}_n$ $\because$ $\lambda_0 + \lambda_1 x + \ldots + \lambda_n x^n \in \text{Span}(B)$

& $0 = \lambda_0 1 + \lambda_1 x + \ldots + \lambda_n x^n$

$\implies \lambda_0 = \lambda_1 = \ldots = \lambda_n = 0$

so B is also lin indep. Hence B is a basis.

e.g. 3 $V = M_{mn}(\mathbb{F})$.

For $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$

define $E_{ij} =$

i.e. 0 everywhere but in (i, j)-th entry which is 1.

$B = \{E_{ij}\}$ is a basis. e.g. for M_{22} we can uniquely express

Linearity of coordinates

Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an ordered basis for vect space V / $\mathbb{F}$.

Rem Then any $\mathbf{v} \in V$ has coord

$(x_1, \ldots, x_n)$

where, $\mathbf{v} = x_1 \mathbf{v}_1 + \ldots + x_n \mathbf{v}_n$.

Thm 1 For $\mathbf{u}, \mathbf{w}, \in V$, $\alpha \in \mathbb{F}$:

a. $[\mathbf{u} + \mathbf{w}]_B = [\mathbf{u}]_B + [\mathbf{w}]_B$.

b. $[\alpha\,\mathbf{u}]_B = \alpha[\mathbf{u}]_B$.

N.B. 1. This means fn $V \longrightarrow \mathbb{F}^n : \mathbf{u} \mapsto [\mathbf{u}]_B$ is linear in language of ch.8.

2. Coord allow you to "identify" V with $\mathbb{F}^n$ the same way it allows you to "identify" 3-dim space with $\mathbb{R}^3$.

Proof: Suppose $\mathbf{u} = u_1\,\mathbf{v}_1 + \ldots + u_n\,\mathbf{v}_n$

$\mathbf{w} = w_1\,\mathbf{v}_1 + \ldots + w_n\,\mathbf{v}_n$

a) $[\mathbf{u} + \mathbf{w}]_B =$

$= (u_1 + w_1,$

$= (u_1,$

$=$

b) Sim.

Scholium $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\} \in V$ is lin indep iff

$[S]_B := \{[\mathbf{w}_1]_B, \ldots, [\mathbf{w}_m]_B\}$ i.e. can check lin indep on coords.

Proof: ($\Longrightarrow$ only). Suppose

$\lambda_1[\mathbf{w}_1]_B + \ldots + \lambda_m[\mathbf{w}_m]_B = \mathbf{0}$

so thm 1 $\Longrightarrow$ $[\lambda_1 \mathbf{w}_1 + \ldots + \lambda_m \mathbf{w}_m]_B = \mathbf{0}$.

Only the the zero vector has coords zero so

$\lambda_1 \mathbf{w}_1 + \ldots + \lambda_m \mathbf{w}_m = \mathbf{0}$.

S lin indep $\Longrightarrow$ $0 = \lambda_1 = \ldots = \lambda_m$

so $\{[\mathbf{w}_1]_B, \ldots, [\mathbf{w}_m]_B\}$ is lin indep too.

Example

e.g.4 Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$\mathbf{v}_1 = (1, 0, 1)^T, \mathbf{v}_2 = (0, 1, 0)^T, \mathbf{v}_3 = (-1, 1, 1)^T$.

i) Show B is a basis for $\mathbb{R}^3$.

ii) Find the coords $[\mathbf{v}]_B$ where $\mathbf{v} = (1, 2, 3)^T$.

A i) Requires showing that for any $\mathbf{b} \in \mathbb{R}^3$, there's a unique soln to

$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_3 \mathbf{v}_3 = \mathbf{b}$.

ii) Requires solving

$x_1\,\mathbf{v}_1 + x_2\,\mathbf{v}_2 + x_3\,\mathbf{v}_3 = (1, 2, 3)^T$.

We re-write this lin eqn in matrix form

Solving shows $[\mathbf{v}]_B = \mathbf{x} =$

To answer i), note that the above computation shows that the row-echelon form for the coeff matrix has all 3 columns leading so back substitution always gives a unique soln regardless of what vector $\mathbf{b}$ we augment the matrix with. Thus B is a basis.

e.g.4 cont'd Find the vector $\mathbf{w}$ with coords $(0, 1, 2)^T$ wrt B.

$\mathbf{A}\ \mathbf{w} = 0\,\mathbf{v}_1 + 1\,\mathbf{v}_2 + 2\,\mathbf{v}_3$

$= (0, 1, 0)^T + 2(-1, 1, 1)^T =$

Orthonormal bases

The most useful coord systems have orthogonal axes.

E.g. In $\mathbb{R}^3$

Recall $B \subset \mathbb{R}^m$ is orthonormal (o/n) if the vectors are unit length & pairwise orthogonal.

Propn 1 An o/n set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is lin indep

Proof: If $\lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n = \mathbf{0}$

then for any i we have

$0 = \mathbf{v}_i \,.\mathbf{0} = \mathbf{v}_i \,.(\lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n)$

$= \lambda_1 \mathbf{v}_i \,.\, \mathbf{v}_1 + \ldots + \lambda_i \mathbf{v}_i \,.\, \mathbf{v}_i + \ldots + \lambda_n \mathbf{v}_i \,.\, \mathbf{v}_n$

$= \lambda_i \mathbf{v}_i \,.\, \mathbf{v}_i = \lambda_i.$

Hence all the scalars $\lambda_i = 0$ & the o/n set must be lin indep.

Propn 2 If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an o/n basis of $\mathbb{R}^m$ (see later $m = n$) then

$[\mathbf{v}]_B = (x_1, \ldots, x_n)^T$ where $x_i = \mathbf{v} \,.\, \mathbf{v}_i$.

Proof: None. This is ex 66 of §7.7 of the notes.

Geometric picture is more enlightening.

Suppose $B = \{\mathbf{v}_1, \mathbf{v}_2\} \subset \mathbb{R}^2$ is o/n.

Alternate characterisation of linear depend

e.g. 5 Suppose non-trivial lin reln holds

$2\,\mathbf{v}_1 + 2\,\mathbf{v}_3 - 4\,\mathbf{v}_4 = \mathbf{0}$.

Then we can write each of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$ as a lin combn of the others

e.g. $\mathbf{v}_1 =$

Prop 3 i) If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is lin depend then there's some i s.t. $\mathbf{v}_i$ is a lin combn of the others.

Conversely, if $\mathbf{v}_i$ is a lin combn of the others i.e. $\mathbf{v}_i \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n)$ then S is lin indep.

Proof: just as in above e.g.

($\implies$) Suppose S lin depend so have non-trivial lin reln

$\lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n = \mathbf{0}$.

Pick i with $\lambda_i \neq 0$. Such an i exists since reln is non-trivial. Now re-write $\mathbf{v}_i$ in terms of others

$$\mathbf{v}_i = -\frac{1}{\lambda_i}(\lambda_1 \mathbf{v}_1 + \ldots \lambda_{i-1} \mathbf{v}_{i-1} + \lambda_{i+1} \mathbf{v}_{i+1} \ldots + \lambda_n \mathbf{v}_n).$$

Conversely if

$\mathbf{v}_i = \lambda_1 \mathbf{v}_1 + \ldots \lambda_{i-1} \mathbf{v}_{i-1} + \lambda_{i+1} \mathbf{v}_{i+1} \ldots + \lambda_n \mathbf{v}_n$

then we can re-arrange to get non-trivial reln

$\lambda_1 \mathbf{v}_1 + \ldots \lambda_{i-1} \mathbf{v}_{i-1} - \mathbf{v}_i + \lambda_{i+1} \mathbf{v}_{i+1} \ldots + \lambda_n \mathbf{v}_n = \mathbf{0}$

so S is lin depend.