Lecture 1: Complex Number Basics

Aim Lecture Extend the real number system to complex number system which includes a square root of -1 denoted i.

Complex numbers

(Crash course only, see notes for details). We won't define complex numbers. For us, a complex number z will be a number of the form $a+bi$ where $a, b \in \mathbb{R}$ and i is a new number s.t. $i^2 = 1$.

The expression $a+bi$ is called the *Cartesian form* for z. The set of complex numbers is denoted $\mathbb{C}$.

We can $+, -, \times$ complex numbers to get a complex number. Below $a, a', b, b' \in \mathbb{R}$.

Add $(1+2i)+(3+5i) = 4+7i$.

In gen, $(a+bi)+(a'+b'i) =$

Subtrn $(1+2i)-(3+5i)=-2-3i$.

In gen, $(a+bi)-(a'+b'i)=$

Multn $(1+2i)(3+4i)=1(3+4i)+2i(3+4i)$

$=3+4i+6i+8i^2=3+10i-8=-5+10i$.

In gen, $(a+bi)(a'+b'i)=$

Rem (for former MATH1081 students) The best defn of a complex number is an equivalence class of real polynomials in i where $p(i)\sim q(i)$ iff $(i^2+1)|p(i)-q(i)$.

How's $\mathbb{C}$ a number system?

Any $x,y,z\in\mathbb{C}$ obey following standard laws of arithmetic.

1. Associative Laws:

$(x+y)+z=x+(y+z), \quad (xy)z=$

2. Commutative Laws:

$x + y = y + x, \quad xy$

3. Distributive Law:

$x(y + z) = xy + xz$

Rem 1. $\implies$ it doesn't matter how you bracket if you stick to just adding or just multiplying. You need the brackets when you add and multiply.

Real and imaginary parts

Defn Let $z = a + bi \in \mathbb{C}, a, b \in \mathbb{R}$.

Its real part is Re $z = a$.

Its imaginary part is Im $z =$

e.g. Re $4 - 5i =$

Thm Let $a, a', b, b' \in \mathbb{R}$. If $a + bi = a' + b'i$ then $a = a', b = b'$ i.e. complex numbers are uniquely determined by their real & imaginary parts.

Proof: $a - a' = b'i - bi =$

$(a - a')^2 =$

$\therefore a = a', b = b'$.

Conjugation & Division

Defn Let $z = a + bi \in \mathbb{C}, a, b \in \mathbb{R}$.

Its complex conjugate is $\bar{z} = a - bi$.

e.g. $\overline{5 - i} =$

Formula $z\bar{z} = (a + bi)(a - bi) = a^2 - b^2 i^2$

so $z\bar{z} = a^2 + b^2$.

This is real!!

Division of complex numbers. Trick is to multiply top & bottom by conjugate of the denominator as follows.

$\frac{1+2i}{3+4i} = \frac{1+2i}{3+4i}\frac{3-4i}{3-4i} = \frac{(1+2i)(3-4i)}{(3+4i)(3-4i)}$

$= \frac{3+6i-4i-8i^2}{3^2+4^2} = \frac{11+2i}{25} =$

Properties of conjugation

Formulae For $w, z \in \mathbb{C}$,

1. $\bar{\bar{z}} = z$

2. $\overline{z - w} = \bar{z} - \bar{w}, \quad \overline{z + w} =$

4. $\overline{zw} = \bar{z}\bar{w},$

5. $\text{Re } z = \frac{1}{2}(z + \bar{z}), \quad \text{Im } z = \frac{1}{2i}(z - \bar{z})$

Proof: easy exercise using Cartesian forms e.g. for 5, if $z = a + bi, a, b \in \mathbb{R}$, then

$\frac{1}{2}(z + \bar{z}) =$

Defn A complex number z is real if $\text{Im } z = 0$ i.e. by 5. above, $z = \bar{z}$. It is *purely imaginary* if $\text{Re } z = 0$ i.e.

e.g. Show that $u = \bar{z}w + z\bar{w}$ is real.

A $\bar{u} =$

How's $\mathbb{C}$ unlike the real number system?

The set P of positive real numbers can be used to order $\mathbb{R}$. P satisfies the following.

i) Any $x \in \mathbb{R}$ satisfies exactly one of the following:

a) $x = 0$ OR b) $x \in P$ OR c) $-x \in P$.

ii) P is closed under addition i.e.

for any $x, y \in P$ we also have $x + y \in P$.

iii) P is closed under multiplication i.e.

for any $x, y \in P$ we also have

We cannot order $\mathbb{C}$ the same way for suppose we can find $P \subset \mathbb{C}$ s.t. ii),iii) hold and i) holds with $\mathbb{R}$ replaced with $\mathbb{C}$. Then either

Argand diagram. Polar form.

Represent $z = a + bi \in \mathbb{C}, a, b \in \mathbb{R}$ by point in plane with co-ordinates (a, b). Above plane called the complex plane & the axes are the real & imaginary axes.

Polar coords on plane suggests

Defn The modulus of $z = a + bi, a, b \in \mathbb{R}$ is

$|z| :=$ distance r from 0 to z

$=$

The argument of $z(\neq 0)$ is the angle

$\operatorname{Arg} z = \theta \in (-\pi, \pi]$ in picture so $\tan\theta =$

Note θ is measured anti-clockwise from the positive real axis so is negative if z lies below the real axis.

e.g.

$|3 + 4i| =$

$\operatorname{Arg} 3 + 4i =$

$|-3 - 4i| =$

$\operatorname{Arg} -3 - 4i =$

Answer here NOT $\tan^{-1}(\frac{-4}{-3})$.

e.g. $|\bar{z}| =$

$\operatorname{Arg} \bar{z} = -\operatorname{Arg} z$ unless

Polar form Consider Cartesian form $z = a + bi$.

If $r = |z|, \theta = \operatorname{Arg} z$ then

$\cos\theta =$

$\implies z = a+bi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

This is called the polar form of z.

e.g. Write $z = 1 + \sqrt{3}$ in polar form.

$|z| =$

z is in the 1st quadrant, so $\operatorname{Arg} z = \tan^{-1}\sqrt{3} =$

The polar form is $z =$