Lecture 9: Existence & Construction of Bases

Aim We show the existence of bases in special cases and illustrate methods of finding them.

Thm 1 Let V = vector space/ field $\mathbb{F}$

Let $S \subset V$ be a spanning set of n elts. Any subspace W

Proof: Omitted. We will prove the special case where $V = \mathbb{R}^3$ below. The generalisation to arbitrary V is easy. The argument below proves the classification of subspaces of $\mathbb{R}^3$ announced in lecture 2.

Let W be a

If $W = 0$, it has basis

Otherwise, let $\mathbf{w}_1 \in$

If $W = \text{Span}(\mathbf{w}_1)$ then W has basis

i.e. W is a

Otherwise, pick $\mathbf{w}_2 \in$

Lemma lecture 8 $\implies$

If $W = \text{Span}(\mathbf{w}_1, \mathbf{w}_2)$ then W has basis

i.e. W is a

otherwise, pick $\mathbf{w}_3 \in$

Lemma lecture 8 $\implies$

But Cor 2 of lecture 8 $\implies$ any 3 lin indep

Hence, $W = \mathbb{R}^3$ & has basis

Above algorithm is useless for finding bases!

Effective Method for Finding Bases

Thm 2 (Reducing spanning sets to bases)

Let $A = (\mathbf{v}_1\,\mathbf{v}_2\ldots\mathbf{v}_n)$. Recall

col$(A) =$

Let U be a

Then we have the following basis of col(A)

$B = \{\mathbf{v}_i$

Proof: will hopefully be clear from following

e.g. (else see notes §6.7.3 p.48)

E.g. 1

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & -3 & 2 & 1 \end{pmatrix} = (\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4)$$

Find a basis for col(A).

Ans:

1st & 2nd

Thm 2 $\implies$

Why did it work?

i.e. why's $\{\mathbf{v}_1$

Check lin indep: Omitting 3rd & 4th column from above calculation, we see

$\therefore$ only soln to $x_1\,\mathbf{v}_1 + x_2\,\mathbf{v}_2 = 0$ is

$\therefore$ $\mathbf{v}_1, \mathbf{v}_2$ are

Check span: 3rd & 4th columns correspond to parameters in soln

Pick soln $\mathbf{x}$ with

Back substn $\Longrightarrow$

$\mathbf{0} = A\,\mathbf{x} =$

Sim, setting

get soln $\mathbf{x} =$

so $\mathbf{v}_4 \in$

$\therefore \text{Span}(\mathbf{v}_1, \mathbf{v}_2) =$

Hence, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is lin indep & spans col(A) so is a basis.

Thm 3 (Completing a lin indep set to a basis)

Suppose W is a subspace of $\mathbb{F}^m$ and $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset W$

There exists a basis B of

Proof: In fact we have

Algorithm for finding B

Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_r\}$ be

Consider the matrix

$A =$

Note $\mathrm{col}(A) \supseteq$

so

Hence we can apply the method of thm 2 to

Note: $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ lin indep $\implies$

$\therefore$ the basis B produced by this method

Hopefully, the reason why this works will be clear from the following e.g.

E.g. 2 Let $S = \{\mathbf{v}_1 = (1, 2, -1)^T, \mathbf{v}_2 = (3, 2, -1)^T\}$. Extend S to a basis of $\mathbb{R}^3$.

Ans: Note vectors not parallel $\Longrightarrow$

Let $\mathbf{w}_1$,

1st, 2nd & 4th

i.e.

Remark What happens to this method if S is lin dependent so that S cannot form part of a basis?

Ans: Method produces a basis with as many members of S as possible.

Finding bases in $M_{mn}(\mathbb{F})$& $\mathbb{P}_k$

We reduce to the $\mathbb{F}^n$ case by the next two lemmas.

Lemma 1 Let V = vector space/ field $\mathbb{F}$

Let B be a basis with n elements.

Let $W \leq$

Define $[W]_B = \{$

Then $[W]_B \leq$

Proof: Just check subspace thm-defn. e.g.

for $\mathbf{w}_1, \mathbf{w}_2 \in$

closure under addn given by

E.g. 3 Let $V = \mathbb{P}_2$ & $B = \{1, x,$

Let $W = \text{Span}(1, x) = \{$

$[W]_B = \{[$

$= \text{Span}($

Lemma 2 Let V, B, W be as above. Let $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\} \subset W$. Then

a) S is lin indep iff

b) S spans W iff

Proof: We'll just do $\implies$ half of b).

$\text{Span}([\mathbf{w}_1]_B, \ldots, [\mathbf{w}_m]_B) =$

$\{\lambda_1$

$= [\mathrm{Span}(S)]_B = [W]_B.$

E.g. 4 From e.g.1,3 above & lemma 2 we see

$\mathrm{Span}(1+x+x^2, -1+x-3x^2, 1+2x^2, 2+3x+x^2)$ has basis

More Results on Dim

Propn Let W be a subspace of

If $\dim V = \dim W$ then

E.g. 3 The only 10-dim subspace of

Proof Propn: Let B be a basis for W.

B is

Cor 1 lecture 8 $\implies$

$\therefore$