Lecture 8: Basis and Dimension

More Examples of Bases

From thm lecture 6 we know

Thm 1 Let V = vector space/ field $\mathbb{F}$. $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis iff every $\mathbf{v} \in V$ is uniquely

E.g. 1 $V = M_{mn}(\mathbb{F})$. For $i \in$

define $E_{ij} =$

i.e. 0 everywhere but

$B = \{E_{ij}\}$ is a basis. e.g. for M_{22} we can uniquely express

E.g. 2 W = line through origin in dirn $\mathbf{w} \in \mathbb{R}^3$.

E.g. 3 W = plane through origin

Questions 1. In the $\mathbb{R}^m$ examples, looks like the number of elements in a basis correspond to the dimension of a subspace. Can use this as a defn

2. Does any vector space have a basis and if so

Aim lectures 8/9: Answer 1 in affirmative and provide effective answer for 2.

More precise answer to 1 suggested by

E.g. 3 revisited W = plane through origin cont'd

More generally have

Thm 2 Let V = vector space/ field $\mathbb{F}$

Let $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$

Let $I = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$

Then

Proof: Omitted. See §6.7.2 of notes. See e.g. 4 below for basic idea of proof.

Thm 3 - Defn Let V = vector space/ field $\mathbb{F}$

Suppose $B_1 = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ and $B_2 = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ are bases for V. Then

In this case, say V

& $\dim V :=$

When no such basis exists,

Proof: By thm 1

B_1 spans V & B_2 lin indep $\implies$

so $m = n$.

E.g. 4 We have a standard basis $\{$

so $\dim \mathbb{R}^n =$

Thm 3 $\implies$ any basis of $\mathbb{R}^n$

We'll prove this without using thms 2 or 3.

Proof: Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be

& $A =$ so A is $n \times m$

We wish to show $m =$

B lin indep $\implies$ $A\,\mathbf{x} = \mathbf{0}$ has

Gaussian elim $\implies$

(otherwise there are more

B spans so $A\,\mathbf{x} = \mathbf{b}$ always

This suggests

Proof is a little delicate. Consider solns $\mathbf{x}_i$

to $A\,\mathbf{x}_i = \mathbf{e}_i$. Let X be $m \times n$-matrix

$AX = (A\,\mathbf{x}_1$

If $m < n$ then above argument $\implies$ there's a non-zero soln $\mathbf{w}$ to $X\,\mathbf{w} =$

But $\mathbf{w} = I\,\mathbf{w} =$

Contradiction $\implies$

This proves thm 3 in this e.g.

E.g. 5

$\dim M_{mn}(\mathbb{F})$

$\dim \mathbb{P}_k$

Corollary 1 For V = vector space/ field $\mathbb{F}$ of dim d

a) Any lin indep

b) Any spanning set

E.g. 6 Can't span $\mathbb{R}^3$ with < 3 vectors and any 4 vectors in $\mathbb{R}^3$ are lin dependent.

We need a lemma on extending lin indep sets

Lemma Let $V =$ vector space/ field $\mathbb{F}$

Let I be a lin indep set. If $\mathbf{v} \in V-$

then $I\cup$

Geom clear as in

Proof lemma: Consider a lin reln

$\lambda\,\mathbf{v} +$

If $\lambda \neq 0$ then solving

Hence,

I lin indep $\implies$

Hence

Corollary 2 Let $V =$ vector space/ field $\mathbb{F}$. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$. The following conditions on B are equivalent:

a) B is a basis

b) B is lin indep &

c)

Proof: a) $\implies$ b) and c) by

Check b) $\implies$ a): Suppose a) false so B doesn't span.

Pick $\mathbf{v} \in$

Lemma $\implies \{\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is

This contradicts Cor 1a) so a) must hold.

Check c) $\implies$ a): Suppose a) false so

Say $\mathbf{v}_1 \in$

E.g. 7 Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an o/n set of vectors in $\mathbb{R}^n$.

Remark Concept of basis for infinite dimensional vector spaces is not as useful. In particular, the defn of span of an infinite set is usually modified.