Lecture 7: Basis and Coordinates

Aim of Lecture: Develop notion of coordinate systems and the concept of bases.

V = vector space/ field $\mathbb{F}$

Defn A subset $B \subset V$ is a basis for V if

a.

& b.

E.g. 1 $V = \mathbb{P}_k$ has standard basis

$B =$

Check: Span $B =$

B is lin indep by

If B is finite, we usually fix an ordering of the elements $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

This is called an

Defn Consider an ordered basis $B =$

$\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of a vector space V over $\mathbb{F}$.

If $\mathbf{v} = x_1\,\mathbf{v}_1 + \ldots + x_n\,\mathbf{v}_n$

then the coordinate vector of

Rem: This is well-defined by thm lecture 6.

e.g.2 Let $B = \{\mathbf{v}_1 = \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right), \mathbf{v}_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$. B spans $\mathbb{R}^2$ & is lin indep $\because$

So B is a basis. The coord vector (wrt B) of $\left(\begin{smallmatrix}1\\4\end{smallmatrix}\right) =$

is

The vector with coord vector $\binom{1}{-1}$ wrt B is

Testing for Bases & Finding Coordinates in $\mathbb{F}^m$

Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ & A be the matrix

Propn B is a basis for $\mathbb{F}^m$ iff

a.

& b.

OR equivalently,

Since $A\mathbf{x} = x_1\mathbf{v}_1 +$

the unique soln is

Stupid E.g. 3 Recall (from lecture 4) that the set of standard basis vectors $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_m\} \in \mathbb{F}^m$ spans $\mathbb{F}^m$.

Here $A =$. You can always solve

i.e. the coord vector of $\mathbf{b}$ wrt standard basis is

E.g. 4 $B = \{\mathbf{v}_1 = (1,0,1)^T, \mathbf{v}_2 = (2,2,-2)^T, \mathbf{v}_3 = (1,1,1)^T\}$.

Is B a basis? If so determine the coordinates of $(2,0,0)^T$ wrt B.

A i.e. we need to ask, does $A\mathbf{x} = \mathbf{b}$ always have a unique soln & if so what's

$(A|\,\mathbf{b}) =$

A is

$\therefore B$

For $\mathbf{b} = (2, 0, 0)^T$,

Testing for Basis & Finding Coordi-

nates in $M_{mn}(\mathbb{F})$ and $\mathbb{P}_k$

As usual, reduce to linear algebra questions of the form considered in the $\mathbb{F}^m$ case, i.e. does $A\,\mathbf{x} = \mathbf{b}$ always have a unique soln and if so, what is the soln?

E.g. 5

$$\mathbf{v}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

Is $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$

If so, what's the coord vector of $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ wrt B?

Ans: B is a basis iff

(*)

always has a

LHS of (*) =

Solving

which always has a unique soln.

$\therefore$

If $\mathbf{b} =$

then

E.g. 1 revisited $B =$

Coordinates of $\lambda_0 + \lambda_1 x + \ldots + \lambda_k x^k$ wrt B is

In this way, we can identify $\mathbb{P}_k$ with $\mathbb{R}^{k+1}$.

Note: In fact, addn and scalar multn also identified e.g.

$[(\sum \lambda_i x^i) + (\sum \mu_i x^i)]_B =$

cont'd

More generally,

Thm Let V = vector space/ field $\mathbb{F}$ and $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an ordered basis. Then for $\mathbf{v}, \mathbf{w}, \in V$, $\alpha \in \mathbb{F}$:

a.

b.

Proof: Sim to e.g. above, see notes §6.8

e.g. 4 revisited Let B be the ordered basis in e.g. 4.

Then $\left[\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\right]_B =$

Orthonormal Bases

The most useful coord systems have orthogonal axes.

E.g. In $\mathbb{R}^3$

Recall $B \subset \mathbb{R}^m$ is orthonormal (o/n) if the vectors are

Propn 1 An o/n set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is lin indep

Proof: If

then for any i

Propn 2 If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an o/n basis of $\mathbb{R}^m$ (see later $m = n$) then

Proof: None. This is ex 60 of §6.8 of the notes.

Geometric picture is more enlightening.

Suppose $B = \{\mathbf{v}_1, \mathbf{v}_2\} \subset \mathbb{R}^2$ is o/n.