Lecture 5: Linear Dependence

Thought Experiment

Study Span(S) inductively by adding one vector at a time to S.

Start with non-zero $\mathbf{w}_1 \in \mathbb{R}^m$ and line

Let $\mathbf{w}_2 \in \mathbb{R}^m$.

Usually (A) Span($\mathbf{w}_1, \mathbf{w}_2$)

unless (B)

Suppose in case (A) above. Let $\mathbf{w}_3 \in \mathbb{R}^m$.
Usually (A) Span$(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$

unless (B)

Upshot: Looks like Span$(\mathbf{w}_1, \ldots, \mathbf{w}_n)$ collapses (as in (B)) and fails to span an n-dim space if

Aim Lectures 5/6: make these ideas precise in notion of linear dependence.

Defn V = vector space/ field $\mathbb{F}$, $S \subseteq V$.

S is linearly dependent if there are

distinct

scalars

with

i.e. some non-trivial linear combn of distinct vectors is zero.

Propn 1 (Alternate charactern of lin depend)

In above case, there is some i with

Conversely, if

Hopefully proof will be clear from following e.g. (else see Notes §6.6.4).

E.g. 1 Let $W \subseteq \mathbb{R}^3$ be subspace consisting of a plane through the origin.

$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in W$

$\therefore \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly

To obtain a non-trivial lin combn as in the defn note

Conversely, from this lin relation we obtain

$\mathbf{v}_1 =$

$\mathbf{v}_2 =$

Defn $S \subseteq V$ = vector space/ field $\mathbb{F}$. Say S is linearly independent if it is not lin dependent.

i.e. If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ (with $\mathbf{v}_i$'s distinct) then the only soln to

Testing lin independence in $\mathbb{F}^m$

As in lecture 4, consider vectors in $\mathbb{F}^m$

$$\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, \mathbf{a}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

cont'd

Saw $x_1 \mathbf{a}_1 + \ldots x_n \mathbf{a}_n = A\mathbf{x}$ hence,

Propn $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is lin independent iff $A\mathbf{x} = \mathbf{0}$ has unique soln $\mathbf{x} = \mathbf{0}$.

E.g. 2 $\mathbf{v}_1 = (1,1,2)^T, \mathbf{v}_2 = (2,1,1)^T, \mathbf{v}_3 = (1,2,5)^T, \mathbf{v}_4 = (0,0,1)^T$

Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ lin independent?

We solve $A\mathbf{x} = \mathbf{0}$.

Third column not leading so

Propn $\Longrightarrow$

In fact $\mathbf{x} = (-3, 1, 1, 0)^T$ is a non-zero soln to $A\,\mathbf{x} = \mathbf{0}$ which corresponds to

the non-trivial relation

From this we also see

$\mathbf{v}_1 =$

$\mathbf{v}_2 =$

But $\mathbf{v}_4$

In fact, above calculation shows $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ lin

Why? Omitting 3rd column from above calculation

all columns

Testing Lin Depend in $M_{mn}(\mathbb{F})$ and $\mathbb{P}$

Reduce question to linear algebra question of the type in e.g. 2 as in the following

E.g. 3 Is $\{p_1(x) = 1 + x + 2x^2, p_2(x) = 2 + x + x^2, p_3(x) = 1 + 2x + 5x^2, p_4(x) = x^2\}$ lin independent in $\mathbb{P}$?

i.e. is the soln $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ the unique soln to

(*)

Equate coefficients:

This gives the same system of linear equations as in E.g. 2.

Since non-zero solns to (*) exist

As before, the non-trivial soln $\lambda =$

gives non-trivial

As in e.g. 2