Lecture 4: More on Linear Combns and Span

Matrix Interpretn of Lin Combn & Span in $\mathbb{F}^m$

$$\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, \mathbf{a}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \in \mathbb{F}$$

Propn The lin combn

$$x_1\,\mathbf{a}_1 + \ldots x_n\,\mathbf{a}_n = A\,\mathbf{x}$$

Proof:

$$A\,\mathbf{x} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Corollary Span($\mathbf{a}_1, \dots, \mathbf{a}_n$) consists of vectors $\mathbf{b}$ such that $A\,\mathbf{x} = \mathbf{b}$ has a soln.

Defn For an $m \times n$-matrix A as above, the column space of A, denoted $\mathrm{col}(A)$ is

Span($\mathbf{a}_1$,

Determining Span in $\mathbb{F}^m$

Stupid E.g. 1 Consider what are called the standard basis vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{F}^m$$

Corresponding matrix is $A = I$. You can always solve

E.g. 2 Is $\mathbf{b} = (0, 1, 2)^T$ in $\text{Span}((1, 2, 3)^T, (4, 5, 6)^T)$?

i.e. can you solve $A\,\mathbf{x} = \mathbf{b}$?

Hence we can solve $A\,\mathbf{x} = \mathbf{b}$ so

E.g. 3 Find all $\mathbf{b}$ in $\text{Span}((1, 0, 2)^T, (3, 1, 2)^T)$.

Try to solve

Hence, $A\mathbf{x} = \mathbf{b}$ is solvable iff

Span here is

E.g. 4 Do $\{(1,0,2)^T, (2,1,4)^T, (3,-2,6)^T\}$ span $\mathbb{R}^3$?

Hence, can't always solve $A\mathbf{x} = \mathbf{b}$:

Determining Span in $M_{mn}(\mathbb{F})$ and $\mathbb{P}$.

Reduce questions to solving a system of linear eqns as follows.

E.g. 5

$$\text{Is } \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \in \text{Span}\left(\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 6 & 0 \end{pmatrix}\right)?$$

Equivalently, can you solve

Comparing entries we see we need to solve for $\lambda, \mu \in \mathbb{F}$

Same

E.g. 6 Which polynomials $b_1 + b_2 x + b_3 x^2$ lie in $\mathrm{Span}(1 + 2x^2, 3 + x + 2x^2)$?

i.e. when can you solve

Compare coefficients:

Same

Hence $b_1 + b_2 x + b_3 x^2$ is in span iff

Span in $\mathcal{R}[\mathbb{R}]$ and other vector spaces

No systematic method here unfortunately.

E.g. 7

$$\cos 3x \cos 2x = \frac{1}{2}\cos x + \frac{1}{2}\cos 5x$$

E.g. 8

$$e^{3x} \notin \text{Span}(e^x, e^{2x})$$

Why?

E.g. 9 Let $\mathbb{R}^\infty$ be the vector space

with addition and scalar multiplication defined coordinate-wise. Let $\mathbf{e}_i =$

Then Span($\mathbf{e}_1$,