Lecture 26: Powers of Matrices

Aim lecture: See applic of diag to computing powers of matrices & studying discrete-time systems.

E.g. 1 Consider $T = T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with e-vectors

Let $\mathcal{B} =$

$T^2 \mathbf{v}_1 =$

$T^3\,\mathbf{v}_1 =$

$T^k\,\mathbf{v}_1 =$

It's thus easy to compute powers of T wrt

If D repr

then T^k (wrt $\mathcal{B}$) is repr by

Change of basis result of thm 2 lecture 23

$\Longrightarrow$

$A^k =$

Let's repeat this problem algebraically as opposed to geom.

Propn 1 Suppose $A \in M_{nn}(\mathbb{F})$ can be diag with $A =$

Then for $k \in \mathbb{N}$, $A^k =$

where $D^k =$

Rem In a sense, propn is true for $k \in \mathbb{R}$.

Proof: $A^k =$

We'll compute D^k only in case $n = 2$ since other cases similar. Note so induction on k gives the answer.

E.g. 2 In e.g. 1 of lecture 24 saw

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1}$$

Find A^4.

Ans:

Discrete-Time Systems

This is a discrete version of the system of linear ODEs studied in lecture 25.

E.g. 3 Define

$x_1(k) =$ no. good hobbits in

$x_2(k)$

Suppose each year, quarter of the good hobbits turn evil & half

i.e. $x_1(k+1) =$

Notn We write $\mathbf{x}(k) =$

We can abbrev the system of linear eqns to

$(*) \qquad\qquad \mathbf{x}(k+1) =$

where $A =$

"Defn" An eqn of the form (*) with $A \in M_{nn}(\mathbb{F})$ arbitrary is said to define a discrete-time system. Say $\mathbf{x}(k)$ evolves according to A.

Note $\mathbf{x}(1) =$

$\mathbf{x}(2) =$

$\mathbf{x}(k) =$

Defn A discrete-time system $\mathbf{x}(k+1) = A\,\mathbf{x}(k)$

is said to be a Markov

Note e.g. 3 is a Markov chain.

Propn 2 If $\mathbf{x}(k+1) = A\,\mathbf{x}(k)$ describes a Markov chain then $\lambda = 1$

Proof: Let $\mathbf{a}_1^T, \ldots, \mathbf{a}_n^T$ be the rows of $A =$

By defn sum of rows

$\implies$ sum of col of

$\implies A^T$

$\Longrightarrow$ ker($I-$

$\Longrightarrow$ det($I-$

i.e. 1 is an e-value.

Back to E.g. 3

Recall $A =$

If init popn is $\mathbf{x}(0) = (1000, 2000)^T$, what happens to $\mathbf{x}(k)$ as $k \longrightarrow \infty$?

Ans: Since $\mathbf{x}(k) = A$

we can compute A^k as in e.g.2 to obtain our answer. We'll be a little more sophisticated in our approach.

Compute e-values first (know $\lambda = 1$ is one of them).

det(

$\lambda =$

$\lambda = 1$ e-vector:

$\ker(I - A) =$

Let $\mathbf{v}_1 =$ $\mathbf{v}_2 =$

Write $\mathbf{x}(0) = a\,\mathbf{v}_1 + b\,\mathbf{v}_2$ for

$\mathbf{x}(k) = A^k\,\mathbf{x}(0) =$

$\longrightarrow$

i.e. limit is an e-vector with

To compute which multiple of $\mathbf{v}_1$ it is, note that the total popn of hobbits remains unchanged and this is given by

$x_1(k) + x_2(k) = x_1(0) +$