Lecture 25: Diagonalisation & Diff Eqns

Let $T : V \longrightarrow V$ have an e-basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ & let $L_i = \text{Span}\, \mathbf{v}_i$.

Aim lecture: See how study of T decomposes into study of

E.g. 1 $y_1(t) =$ popn of hobbits in

$y_2(t) =$ popn of orcs

If 2 popn kept separate as here then popn growth governed by a pair of DEs which typically looks something like:

$$\begin{aligned} y_1'(t) &= 3y_1(t) \\ y_2'(t) &= 2y_2(t) \end{aligned} \qquad (*)$$

Soln: Easy, solve 2 eqns <u>separately</u>

Suppose now we put the two popns in

Typical DEs describing popn growth is

$$\begin{aligned} y_1'(t) &= 3y_1(t) - 2y_2(t) \\ y_2'(t) &= -y_1(t) + 2y_2(t) \end{aligned} \qquad (\dagger)$$

These are "coupled" DEs i.e. y_1', y_2' each depend on both $y_1 \& y_2$. We'll use diag to

Notn $\mathbf{y}(t) =$

$\mathbf{y}'(t) = \frac{d\,\mathbf{y}}{dt} :=$

In e.g. 1, $\mathbf{y}'(t) = A\,\mathbf{y}(t)$ where

$A =$

Note In decoupled case (*) above, still have $\mathbf{y}'(t) = A\,\mathbf{y}(t)$ but now

Change of Var & DEs

Consider more generally $\mathbf{y}(t) = (y_1(t),$

& system of n linear DEs

$\mathbf{y}'(t) = A\,\mathbf{y}(t)$

where $A \in M_{n,n}(\mathbb{R})$.

Lemma For $C \in M_{n,n}(\mathbb{R})$

$\frac{d}{dt}$

Proof: Clear from case $n = 2$. Say

$C =$

$\frac{d}{dt}(C\,\mathbf{y}) =$

Back to solving $\mathbf{y}'(t) = A\,\mathbf{y}(t)$, suppose we can diag $A = MDM^{-1}$ with

$D =$

$MDM^{-1}\,\mathbf{y} =$

$\therefore\, DM^{-1}$

Let's change var to $\mathbf{x}(t) =$

See $\frac{d\,\mathbf{x}}{dt}$

This is decoupled. Can solve the n linear DEs

to get $x_i(t) = \alpha_i$

Upshot The soln to $\mathbf{y}'(t) = A\,\mathbf{y}(t)$ is

$\mathbf{y}(t) = M\,\mathbf{x}(t) =$

$=$

where $\lambda_i =$

& $\mathbf{f}_i =$ corresp e-vectors.

E.g. 1 completed

We diag A

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 2 \\ 1 & \lambda - 2 \end{vmatrix}$$

The e-values are

E-vectors?

$\lambda = 4 : \ker(\lambda I - A) =$

An e-vector is

$\lambda = 1 : \ker(\lambda I - A) =$

An e-vector is

Hence (from upshot) soln given by

$\mathbf{y}(t) =$

i.e. $y_1(t) =$

E.g. 2 Suppose in e.g. 1 that initial popn is $\mathbf{y}(0)^T = (4000, 1000)$. Solve the IVP.

Ans: We need only solve for α_1, α_2.

From Gaussian elim or guessing see

$\alpha_1 =$

The soln is thus

$\mathbf{y}(t) =$

2nd Order ODEs

We can convert any 2nd order ODE into a pair of linear ODEs in 2 var as in following

E.g. 3 Solve IVP

$$y'' - 3y' + 2y = 0 \quad , \quad y(0) = 2, y'(0) = 3$$

Ans: Let $y_1 = y, y_2 =$

$y_1' =$

i.e. $\mathbf{y}' =$

Diag A: $\det(\lambda I - A) =$

Hence e-values are

E-vectors:

$\lambda = 2 : \ker(\lambda I - A) =$

An e-vector is

$\lambda = 1 : \ker(\lambda I - A) =$

An e-vector is

Hence, (from upshot) general soln is

$\mathbf{y}(t) =$

Need now find integration constants.