Lecture 24: Finding Eigenbases

Consider $T : V \longrightarrow$

Examined last time certain preferred bases for

E.g.

Study of T breaks down to studying

Q 1. How do you find these preferred bases?

2. How do these bases improve our under-

standing of T & simplify

Aim this lecture: Answer Q1.

Aim next lecture: Answer Q2.

Re-interpret E-vectors

Propn 1 Let $T : V \longrightarrow V$ lin map

$\mathbf{v} \in V$ is an e-vector with e-value λ iff $\mathbf{v} \in$

Proof:

Finding Eigenbases

Let $T : V \longrightarrow V$ lin map

Propn $\implies$ if you know the e-values then the e-vectors

Hence consider first

Finding Eigenvalues

Let $T = T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ lin map

Propn 1 tells us, λ is an e-value of T (or A)

iff

iff

iff det

This suggests

Propn-Defn Let $A \in M_{n,n}(\mathbb{F})$. The polynomial (in λ)

$p(\lambda) :=$

is called the characteristic polynomial of A. It has degree n and its roots are

Proof: Above argument shows the roots coincide

We'll only check $\deg p$ when

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Gen case uses induction (really boring).

N.B. To ensure $p(\lambda)$ has roots, often advantageous to work with $\mathbb{F} = \mathbb{C}$.

E.g. 1 Find e-values & e-vectors of

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$$

Ans:

Step 1: Find roots of $p(\lambda) =$

Step 2: Find corresponding e-vectors

For $\lambda = 2$: $\ker(\lambda I - A) = \ker$

cont'd

$\therefore \alpha \binom{1}{1}$

For $\lambda = -1$: $\ker(\lambda I - A) =$

$\therefore \alpha \binom{2}{1}$

Recall change of basis result (thm 2) from last lecture:

Let A be an $n \times n$-matrix so $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ & $\mathcal{B} = \{\mathbf{f}_1, \ldots \mathbf{f}_n\}$ be a basis of $\mathbb{F}^n$. The matrix C rep T_A wrt $\mathcal{B}$ is

$C = M^{-1}AM$ where $M = (\mathbf{f}_1 \ldots \mathbf{f}_n)$.

E.g. 1 cont'd Here we have an e-basis for A viz.

$\mathcal{B} := \{\mathbf{f}_1 =$

Thm 1 lecture 23 $\Longrightarrow$ matrix rep T_A wrt $\mathcal{B}$ is diag matrix

Change of basis shows then

$M^{-1}AM =$

for $M =$

i.e. $A =$

Defn (Diagonalisable) $A \in M_{n,n}(\mathbb{F})$ is diagonalisable (over $\mathbb{F}$) if there's a diag matrix

$D =$

& an invertible $n \times n$-matrix M s.t.

Finding such an expression is called diagonalising A.

Thm Let $A \in M_{n,n}(F)$. A can be diagonalised with $A = MDM^{-1}$ iff

i)the columns $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ of M form an

& ii) D has diag entries the

Proof: As in last e.g., thms 1 & 2 of last lecture show if A has an e-basis as above then

Conversely, suppose $A = MDM^{-1}$ where D is diagonal with entries $\lambda_1, \ldots, \lambda_n$.

M invertible $\implies \{\mathbf{f}_1,$

Expand $AM = MD$ to get

$$A(\mathbf{f}_1 \ldots \mathbf{f}_n) \qquad = \qquad (\mathbf{f}_1 \ldots \mathbf{f}_n)$$

$$(A\,\mathbf{f}_1 \ldots A\,\mathbf{f}_n) \qquad =$$

Equate columns to see $\mathbf{f}_i$ is an

E.g. 2 Diagonalise

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -4 & 6 \\ 0 & -3 & 5 \end{pmatrix}$$

Note, if A as in e.g. 1, then $B =$

$$\det(\lambda I - B) = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda + 4 & -6 \\ 0 & 3 & \lambda - 5 \end{vmatrix} =$$

So e-values

E-vectors:

$\lambda = 1$:

$$\ker(\lambda I - B) = \ker \begin{pmatrix} & -1 & 0 \\ 0 & & -6 \\ 0 & 3 & \end{pmatrix}$$

An e-vector with e-value 1 is

$\lambda = 2$:

$$\ker(\lambda I - B) = \ker \begin{pmatrix} & -1 & 0 \\ 0 & & -6 \\ 0 & 3 & \end{pmatrix}$$

An e-vector with e-value 2 is

$\lambda = -1$:

$$\ker(\lambda I - B) = \ker \begin{pmatrix} & -1 & 0 \\ 0 & & -6 \\ 0 & 3 & \end{pmatrix}$$

An e-vector with e-value -1 is

If $M =$

then $A =$

Q 1. Do e-bases always exist?

2. Do e-vectors always exist?

Ans Q2: In case of a matrix A, this depends on roots of $\det(\lambda I - A)$ which always exist if $\mathbb{F} = \mathbb{C}$. However, if in $T : V \longrightarrow V$, we have $\dim V = \infty$ then there may be no e-vectors even if $\mathbb{F} = \mathbb{C}$.

E.g. 3 $V = \mathbb{C}^{\infty}$

Let $T : V \longrightarrow V$ be shift operator

T has no e-vectors.

Why? Suppose $(a_0, a_1, \ldots)$ is an e-vector so that

If $\lambda = 0$ then

If $\lambda \neq 0$ then comparing 1st, 2nd, 3rd, etc coords we see,

$\therefore$

Ans Q1: No even if $\mathbb{F} = \mathbb{C}, V = \mathbb{C}^2$.

E.g. 4 Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} \lambda - 1 & 1 \\ 0 & \lambda - 1 \end{vmatrix}$$

so the only e-value is

$$\ker(I - A) = \ker \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =$$

No e-basis as all e-vectors span 1-dim space only.

In 2nd yr lin algebra you'll study linear maps like these in greater depth. The only result in this course guaranteeing e-bases is

Propn 2 If $A \in M_{n,n}(\mathbb{F})$ has n distinct e-values then A is diagonalisable, i.e. has an e-basis.

Proof: Let e-values be $\lambda_1, \ldots, \lambda_n$ & the corresp

Suffice show $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is lin indep & $\therefore$

a

Suppose not so

$\alpha_1 \mathbf{v}_1 +$

some α_i

Rearranging indices, we can assume

$\alpha_1 \mathbf{v}_1 +$

with $\alpha_1 \neq 0, \ldots, \alpha_i \neq 0$ & i is

$\mathbf{0} = (T - \lambda_i I)(\alpha_1 \mathbf{v}_1 +$

$= \alpha_1 (T - \lambda_i I) \mathbf{v}_1 +$