Lecture 18: Continuous Random Variables

A continuous random var $X : \Omega \longrightarrow \mathbb{R}$ can take any value in $\mathbb{R}$.

E.g. $X =$ humidity in Sydney.

Aim Lecture Set up probability theory for these random var.

Approach

Consider e.g. above. We'll use

Range of values is

We'll divide this into 4

of length Δ

Let $p_i =$

Draw normalised graph of $p'_k :=$

We want to shrink $\Delta \longrightarrow 0$ & will see later that this normalisation removes dependence on Δ.

Key Point

$P(X$

Typical graph of p'_k for $\Delta =$

Let's subdivide

Note: Each original rectangle breaks up into

What Happens as $\Delta \longrightarrow 0$?

1. $p_k =$ area

$\therefore P(X = x)$

2. The graph of p'_k tends towards a presumably cont fn $f(x)$. Prob is represented by

3. Since $\sum p_k = 1, p_k \geq 0$ we expect

cont'd

Above heuristics suggest

Defn (Continuous Random Var)

A random var $X : \Omega \longrightarrow \mathbb{R}$ is continuous if there's an

which gives the probability

$P(a <$

such that

1.

2.

We call $f(x)$ the probability density fn or

Any fn $f(x)$ satisfying 1 & 2 is called a

N.B. $P(X = a) =$

so $P(a \leq X < b) =$

E.g. 1 $X =$ time (in hrs) Homer Simpson

Suppose prob density fn $f(x)$ is proportional to x. What's $f(x)$?

Ans: $f(x) =$

Need only determine

Negative Exponential Distribution

E.g. 2 Let $\alpha > 0$ be a parameter.

Define $f(x)$

$f(x)$ is a prob density fn. Why?

1.

2. $\int_{-\infty}^{\infty} f(x)dx =$

It's graph

Note 1 & 2 means we usually have $f(x) \longrightarrow$

Cumulative Distrbn Fn

Given a probability density fn $f(x)$ for random var X, it's sometimes easier to work with actual probabilities.

Defn The cumulative distrbn fn of $f(x)$ is

$F(x) :=$

E.g. 2 again Negative exp distrbn

For $x \leq 0, F(x) =$

If $x \geq 0$, $F(x) =$

Graph

Can compute the median value from this i.e. the value of x such that

$.5 = P(X$

$e^{-\alpha x} =$

Median $x =$

Note 1. If f is cont, $F'(x) =$

2. $f(x) \geq 0 \implies$

3. $\int_{-\infty}^{\infty} f(x)dx = 1 \implies$

4. $P(a < X < b) =$

Mean (Cont case)

Again use discrete approx to suggest defn.

Recall

$p'_k = \frac{p_k}{\Delta} \longrightarrow$

Usual limiting argument shows the mean in discrete approx

$\sum x_k p_k =$

Suggests

Defn (Expected Value)

Let $f(x)$ be prob density fn for cont random var X. The

$E(X) :=$

E.g. 2 again $f(x)$ negative exp distrbn with param α.

$E(X) =$

Formulae Involving Expected Values

As in discrete case, given a continuous fn $g : \mathbb{R} \longrightarrow \mathbb{R}$, we can consider new random var $g(X)$.

Propn 1 If X is a cont random var with prob density $f(x)$ then

$E(g(X)) =$

Proof: See ex 28, ch. 9 of the notes.

Propn 2 a) $E(g(X) + h(X)) =$

b) For $a, b \in \mathbb{R}$, $E(aX + b) =$

Proof: Use propn 1 in both cases.

a)

b) Sim (see discrete case for hints).

Variance (Cont case)

Let $f(x)$ be prob density fn for random var X. The

Var(X) :=

Its square root $\sigma(X)$ is

Propn 3 $Var(X) =$

Proof: Same as in discrete case. Use propn 2.

E.g. 2 again Negative exp distrbn fn $f(x)$ with param α.

Recall $f(x) =$

& $E(X) =$

cont'd $Var(X) =$

E.g. 3

$X =$ diameter of Spider

Assume exp distribn with mean 2mm. Find expected value of cross-

Ans: $E($