Lecture 14: Invertible Linear Maps

Aim Lecture Coordinates allow you to identify fin dim vector spaces with

More gen, invertible

Defn (One-to-one) Let X, Y be sets & $f : X \longrightarrow Y$ be a function. We say that f is one-to-one (1-1) or injective if the soln

i.e. $f(x) = f(x')$

E.g. 1 $f(x) = x^2$ is not

Notion of 1-1 simplifies in the linear case.

Propn 2 Let $T : V \longrightarrow W$ be linear.

Then T is 1-1 iff

Proof: Recall from lecture 12 propn 2 that given $\mathbf{w} \in W$ & particular

$\therefore$ soln is unique

E.g. 2 Show the soln to the IVP

$$\frac{dy}{dx} - y = \frac{\pi^x}{\sqrt{e^{\cos x} + \cosh x^{\pi}}}, y(0) = 0$$

is unique?

Ans: Let T be defined by

Note T is linear being

By propn, suffice show $\ker T = \mathbf{0}$

i.e. (*)

has unique soln $y = 0$.

Defn (Onto) A fn $f : X \longrightarrow Y$ is

i.e. im f

Inverse Functions

Let X, Y be sets and $f : X \longrightarrow Y$ any fn.

Propn-Defn The following condns on f are equivalent.

a) f is 1-1 &

b) the eqn $f(x) = y$ always has a

denoted $x =$

c) there's a fn g :

Proof: Clear a) & b) equiv

If b) holds, then c) holds on

Conversely, if $g : Y \longrightarrow X$ is as in c), then

$x = g(y)$ is

Check is a soln: $f(g(y))$

Unique: if also $f(x') = y =$

$x' = g($

Propn 3 If $T : V \longrightarrow W$ is linear and invertible then

Proof: in case $T = T_A$ for $A \in M_{mn}(\mathbb{F})$.

Since T is invertible, $T\,\mathbf{x} = A\,\mathbf{x} = \mathbf{b}$ always

Hence, $T^{-1} =$

E.g. 3 Let V = vector space/ field $\mathbb{F}$

$B = \{\mathbf{v}_1,$

Define $S_B : V \longrightarrow \mathbb{F}^n$ by

Recall $S_B(x_1\,\mathbf{v}_1 + \ldots + x_n\,\mathbf{v}_n) =$

Thm lecture 7 $\implies$

S_B is invertible since $S_B(\mathbf{v}) = (x_1, \ldots, x_n)^T$

has unique soln

Part a) of following propn generalises fact that a subspace has smaller dimension than the ambient space.

Propn 4 Let $T : V \longrightarrow W$ be linear.

Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset$

a) T 1-1 & B lin indep $\implies$

Hence if W is finite dim then so is

b) T onto & B a spanning set $\implies$

Proof: a) We'll show $T(B)$ is

Suppose $\mathbf{0} =$

Need all $\lambda_i = 0$.

$\mathbf{0} =$

$\therefore T$ 1-1 $\implies$

B lin indep

Hence $T(B)$ is

If B is a basis, then lecture 8 cor 1 $\implies$

b) Sim, see proof lemma 2 lecture 9.

Cor If $T : V \longrightarrow W$ is an invertible

Then bases (resp. spanning sets, resp. lin indep sets) of V & W correspond via T & T^{-1}.

E.g. 4 Let $B := \{\mathbf{v}, \mathbf{w}\}$ be a basis for V. Then $\{\mathbf{v} + \mathbf{w}, \mathbf{v} -$

Why? Consider invertible lin map S_B :

Hence $\{\mathbf{v} + \mathbf{w}$

Propn 5 Let $T : V \longrightarrow W$ be a lin map

Consider invertible

$S_r : V' \longrightarrow V, \quad S_l$

$S_l T S_r :$

a) $\ker(S_l T S_r) =$

b) $\mathrm{im}\,(S_l T S_r) =$

Proof: a) $\mathbf{x} \in \ker(S_l T S_r)$ iff

iff

b) $S_l T S_r(V') =$

We can now prove propn 3, lecture 12 & propn 1, lecture 13.

Cor Let $T : V \longrightarrow W$ be

of dimensions n & m.

& B_V, B_W be finite ordered

Let A be the matrix

a) The corresp map on coord vectors is

$T_A =$

b) $[\ker T]_{B_V} =$

c) $[\text{im } T]_{B_W} =$

Proof: a) is an easy ex. In fact, you can prove the gen matrix reprn thm by applying matrix reprn thm to

b) $\ker A =$

c) is almost identical.

A nice application of rank-nullity is the following.

Propn 6 Let $T : V \longrightarrow W$ be linear and suppose $\dim V = \dim W$ is finite.

Then T is invertible iff either

Proof: If T is 1-1: $\text{null}\, T = 0 \implies$

rank $T =$

If T is onto: