Lecture 13: Solving Linear Eqns. Image

Aim Lecture For $T : V \longrightarrow W$ linear, understand when you can solve

Image

Defn-Propn (Image) Let $T : V \longrightarrow W$ be linear. The image of T is

Also rank $T :=$

If $T = T_A$ also write

Note im $A = \text{col}\, A$.

Proof: Follows easily from subspace thm-defn. We'll just check closure under addn. For $\mathbf{v}, \mathbf{w} \in V$, so im T is closed

E.g. 1 $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ orthog projn onto

Finding Bases for Image

For $\mathbb{F}^m$ case, we computed bases for col(A) in thm 2 lecture 9. We reduce to this case using

Propn 1 Let $T : V \longrightarrow W$ be

& B_V, B_W be finite ordered

Let A be the matrix

Then $[\text{im } T]_{B_W} =$

Proof: in lecture 14.

E.g. 2 Let $M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

Define $T : M_{22}(\mathbb{F}) \longrightarrow M_{22}(\mathbb{F})$ by $TC :=$ MC. Note that T is linear by

Find a basis for im T and rank T.

Ans: First find matrix $A \in M_{44}(\mathbb{F})$ representing T wrt

$\mathbf{e}_{11} =$

$$T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence $A = ([T$

$=$

We compute a basis for $\mathrm{col}(A)$ as in lecture 9.

$A \longrightarrow$

Hence a basis is

$\mathbf{v}_1 =$

They correspond to

$C_1 =$

By propn 1 & lemma 2 of lecture 9,

$\{C_1, C_2\}$ is

Rank-Nullity Thm

Thm Let $T : V \longrightarrow W$ be a linear map, V, W finite dim.

Let A be a matrix

wrt

Let U be a row

a) $\operatorname{null} T =$ no. non-leading columns

b) rank $T =$

c) (Rank-Nullity) $\dim V =$ no.

Proof: Do easy case only where $T = T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ where $A = (\mathbf{v}_1 \ldots \mathbf{v}_n)$

a) Each non-leading column in U gives

a param in

$\therefore$ gives a basis

$\therefore$ null $A =$

& a) holds.

b) Thm 2, lecture 9 gives basis

$\{\mathbf{v}_i \,|i$

so b) holds.

c) Add a) & b)

Gen case reduces to this one using methods of lecture 14.

E.g. 3 A geometric picture illustrating rank-nullity thm

Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be an orthog projn onto 1-dim subspace W.

E.g. 4 We'll use the rank-nullity thm to show that the intersection of m hyperplanes through $\mathbf{0}$ in $\mathbb{R}^n$ has

For $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n - \mathbf{0}$ consider the hyperplanes

$H_i :=$

Now $\mathbf{v} \,.\, \mathbf{v}_i = \mathbf{v}_i^T \mathbf{v} \implies H_i =$

Hence, $\cap H_i = \cap \ker \mathbf{v}_i^T =$

Now im A is a sub

so rank $A =$

Hence $\dim \cap H_i = \dim \ker A =$

E.g. 5 Let $A \in M_{nn}$. Define fn $T : M_{nn} \longrightarrow M_{nn}$ by

$TB := AB-$

Show im $T \neq$

Ans Check 1st T is linear. If $B, C \in$

$T(B + C) =$

Also $T(\lambda B) =$

So T is

If $A = 0$ then im $T = 0$ so suppose

so $A \in$

$\dim \operatorname{im} T =$