Lecture 12: Solving Linear Eqns in Arbitrary Vector Spaces

E.g. 1 Multiplication by a function is linear. For a real-valued fn $g(x)$ on $\mathbb{R}$, define $T : \mathcal{R}[\mathbb{R}] \longrightarrow \mathcal{R}[\mathbb{R}]$ by

Then T is lin by the distrib &

E.g. 2 Define $T : \mathbb{P}_2(\mathbb{R}) \longrightarrow \mathbb{P}_3(\mathbb{R})$ by

$$Tp = (1+x)\frac{dp}{dx} - 2p$$

Note T is linear being the sum of the lin maps

$-2\mathrm{Id} : p(x) \mapsto$

&

The latter is linear being

cont'd

Q Find the matrix reprn of T with respect to the basis

$B_2 = \{$

$B_3 = \{$

Ans: $A = ([T1]_{B_3}$

$T1 =$

$[T1]_{B_3} =$

$Tx =$

$[Tx]_{B_3} =$

$Tx^2 =$

$[Tx^2]_{B_3} =$

Hence, $A =$

Aim Lecture 12 Let $T : V \longrightarrow W$ be a linear map such as in e.g. 2. Given $\mathbf{w} \in W$ what do solns to the "linear eqn"

Kernels

Defn (Kernel,Nullity)

Let $T : V \longrightarrow W$ be linear. The kernel of T is

$\ker T :=$

i.e. "Homogeneous Solns". See below that kernels are subspaces so can define the nullity of T to be $\operatorname{null} T :=$

E.g. 3 $T = \frac{d}{dx} : \mathbb{P} \longrightarrow \mathbb{P}$ has kernel solns to

$\ker T =$

$\operatorname{null} T =$

Propn 1 Kernel is a Subspace

If $T : V \longrightarrow W$ is linear then $\ker T$ is a

Proof: a) $\ker T$ non-empty as propn 1 lecture 10 $\Longrightarrow$

b) If $\mathbf{v}, \mathbf{v'} \in \ker T$ then

c) If $\lambda \in \mathbb{F}$ then

Subspace thm-defn

Dumb E.g. 4 Show

$W = \{\mathbf{x} \in \mathbb{R}^4 \,| x_1 +$

is a subspace of $\mathbb{R}^4$.

Ans:

The nature of solns to $T\mathbf{v} = \mathbf{w}$ is sim to $\mathbb{F}^m$ case as following propn shows.

Propn 2 Let $T : V \longrightarrow W$ be linear. Suppose given $\mathbf{w} \in W$ and a particular soln

$\mathbf{v} = \mathbf{v}_p$ to eqn

(*)

The complete set of

Proof: As in session 1. Don't believe me?

Observe

$\mathbf{v}_h + \mathbf{v}_p$ is a soln since

If $\mathbf{v}$ is a soln so $T\,\mathbf{v}$

then

$\implies \mathbf{v}_h := \mathbf{v} - \mathbf{v}_p \in$

Hence **v**

Finding Bases for Ker ($\mathbb{F}^m$ Case)

Represent linear $T : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ by matrix

$A \in$

Define $\ker A$ $\operatorname{null} A$

E.g. 5

$$A = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 0 & -1 \end{pmatrix}$$

Find

Ans: Need to solve

If $x_3 = \lambda, x_4 = \mu$ then

$x_1 =$

$\therefore \mathbf{x} =$

Hence $B =$

spans $\ker A$ &

so B is a basis for

$\operatorname{null} A =$

Note The spanning set for $\ker A$ obtained by this procedure is always

Finding Bases for Ker (Gen Case)

Propn 3 Let $T : V \longrightarrow W$ be

& B_V, B_W be finite ordered

Let A be the matrix

Then $[\ker T]_{B_V} =$

Proof: See lecture 14.

Propn allows one to compute $\ker T$ as in

E.g. 2 revisited $T : \mathbb{P}_2(\mathbb{R}) \longrightarrow \mathbb{P}_3(\mathbb{R})$

defined by

$$Tp = (1 + x)\frac{dp}{dx} - 2p$$

What's $\ker T$?

Ans: Recall matrix reprn

A

$B_2 =$

$\ker A$ consists of vectors $\mathbf{v} =$

Propn $\implies$ the $p(x) \in \ker T$ are precisely those whose coord

i.e. $p(x) =$

Hence $\ker T =$

Check: if $p(x) = \lambda(1 + x)^2, p'(x) =$

Hence

Rem Let B be a basis of V. Recall lemma 2 lecture 9, that $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a basis for a subspace W iff

so method above allows you to find

Alternate Method: Solve linear ODE

by separating variables or using an integrating factor.

Geom Viewpoint of Solving Lin Eqn

E.g. 6 Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be orthogonal projn onto 1-dim subspace L