Lecture 11: Geom Examples of Lin Maps

Aim Lecture 11 Exhibit various geometric transformations such as rotations and reflections as linear maps.

Orthogonal Projections

Let $\mathbf{u} \in \mathbb{R}^n$ be

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be projn onto line $L :=$

Recall $T\,\mathbf{v} = (\mathbf{v}\,.$

T is linear.

Why? Add Condn: for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$

$T(\mathbf{v}+\mathbf{w}) =$

$= (\mathbf{v} \,.\, \mathbf{u} +$

$= (\mathbf{v} \,.\, \mathbf{u})\, \mathbf{u} +$

$=$

so addn

Scalar Multn: For $\lambda \in \mathbb{R}$,

Hence T is lin.

We can check these condns geometrically too.

Defn Let $T, T' : V \longrightarrow W$ be

Let $\lambda, \mu \in \mathbb{F}$. Define new maps

Sum: $T + T' : V \longrightarrow W$ by

scalar multiple: $\lambda T : V \longrightarrow W$ by

Linear Combn: $\lambda T + \mu T' : V \longrightarrow W$ by

Propn 1 $T + T', \lambda T$ are linear too. Hence so is $\lambda T + \mu T'$.

Proof: Check λT lin: for $\mathbf{v}, \mathbf{w} \in V, \alpha \in \mathbb{F}$

Add Condn: $(\lambda T)(\mathbf{v} + \mathbf{w}) \stackrel{\text{def}}{=}$

$= \lambda(T\mathbf{v} +$

$= \lambda(T\mathbf{v})$

$=$

so addn condn holds.

Scalar Multn Condn: $(\lambda T)(\alpha\,\mathbf{v}) =$

So scalar multn condn also holds &

Check $T + T'$ linear: Can prove as above. Here we only prove in case $V = \mathbb{F}^n, W = \mathbb{F}^m$ so by matrix representation thm there are $A, B \in M_{mn}(\mathbb{F})$ with

$(T + T')\,\mathbf{v} =$

$T + T'$ is matrix

Proof gives

Formula $T_A + T_B = T_{A+B}$

Sim $\lambda T_A = T_{\lambda A}$.

Identity Linear Map

Let $\mathrm{Id}_n : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ be the function $\mathrm{Id}_n \mathbf{v} =$

$\mathbf{v} =$

Hence Id_n is linear.

Reflections

Let $\mathbf{u} \in \mathbb{R}^n$ be unit vector and $L =$

Let $P : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be projn

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be reflection

Propn 1 $\implies$ $T =$

Rotations by angle $\theta \in$

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be anti-clockwise

What's $T\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$?

Use complex numbers.

$e^{i\theta}(x + iy) =$

$$\therefore T \begin{pmatrix} x \\ \\ y \end{pmatrix} =$$

Hence $T = T_{R_\theta}$ is linear. R_θ is

Composite Functions

Propn 2 Let $T : V \longrightarrow W, S : U \longrightarrow V$ be linear maps. Then $T \circ S$:

Proof: Only do in case $U = \mathbb{F}^n, V = \mathbb{F}^m, W = \mathbb{F}^l$. Matrix rep thm $\Longrightarrow$

For some $A \in$

For some $B \in$

$(T \circ S)\,\mathbf{v} =$

Formula

E.g. 1 Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so $T = T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T_A \circ T_A =$ rotation

$A^2 =$

so T_{A^2} is also

i.e. $T_A \circ T_A = T_{A^2}$.

Propn 3 Let $A \in M_{mn}$. Then T_A takes lines

Proof Consider line L with parametric form

$T_A(L)$ is set of pts $T\,\mathbf{x} =$

This a

Note this holds for geom examples above.

Bases give coords for vector spaces & allow us to "identify" vector spaces with $\mathbb{F}^n$. Using this we can represent lin maps between more general vector spaces by matrices as follows.

Thm (Generalised Matrix Rep Thm)

Let $T : V \longrightarrow W$ be

$B_V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an

$B_W = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be an

Let $A = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ be

with columns

Then $[T\,\mathbf{v}]_{B_W} =$

Proof: Omitted (but see lecture 14). The following e.g. should illuminate the procedure.

E.g. 2 Let $V = W = \mathbb{P}_2$ and $T : \mathbb{P}_2 \longrightarrow \mathbb{P}_2$ be differentiation (which is lin!). Let $B = B_V = B_W :=$

Matrix reprn is $A =$

$\mathbf{a}_1 =$

$\mathbf{a}_2 =$

$\mathbf{a}_3 =$

How does A repr T?

If we identify $p(x) = \lambda_1 + \lambda_2 x + \lambda_3 x^2$ with its coord

$[Tp(x)]_B =$

$= A[$

i.e. corresp map on coord vectors $(\lambda_1,$

The gen matrix repr thm proved by applying usual matrix repr thm to the correspond