Lecture 10: Linear Maps or Transformations

Recall, a (homogeneous) linear function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is one of the form

$f(x_1, \ldots, x_n)^T =$

More generally, a lin fn $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is one of the form

Aim Lecture 10 Generalise the notion of linear functions to linear maps $T : V \longrightarrow W$ between

Defn Let V, W = vector space / $\mathbb{F}$ and $T : V \longrightarrow W$ a function. We say T is a linear map or

a. (Addition Condn) For any $\mathbf{v}, \mathbf{w} \in V$

& b. (Scalar Multn Condn) For any $\lambda \in \mathbb{F}$

E.g. 1 Let $A \in M_{mn}(\mathbb{F})$. Define $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ by

Proof: (Addn condn) for $\mathbf{v}, \mathbf{w} \in V$

$T_A(\mathbf{v} + \mathbf{w}) =$

so addn condn holds

(Scalar Multn Condn) if also $\lambda \in \mathbb{F}$

$T_A(\lambda\,\mathbf{v}) =$

giving scalar

Since the addn & scalar multn condn

E.g. 2 Saruman needs

x_1

x_2

Each orc

Each

If y_1, y_2 are

$\mathbf{y} =$

so $\mathbf{y}$ is a lin fn of

E.g. 3 $\mathcal{C} =$

$\mathcal{C}^1 =$ subspace of continuously

Define $T : \mathcal{C}^1 \longrightarrow \mathcal{C}$ by

T is linear.

Why? Addn: for $f, g \in \mathcal{C}^1$

$T(f + g) =$

so addn condn holds

Scalar Multn: for $\lambda \in \mathbb{R}$

$T(\lambda f) =$

Hence, scalar multn condn also

E.g. 4 Let $T : M_{23}(\mathbb{F}) \longrightarrow M_{32}(\mathbb{F})$ be defined by

T is linear.

Why? Addn: For $A, B \in M_{23}$

$T(A + B) =$

Scalar Multn: for $\lambda \in \mathbb{F}$

$T(\lambda A) =$

Hence, scalar multn condn also

Linear maps are very special. One way to view them is

Fact A function $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear iff its graph is

We'll omit the proof and definition of graph but hopefully the following example shows what's going on.

Non- E.g. 5

$T : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $T(x) = x^2$.

Graph of T

$T(2) =$ but $2T(1) =$

Scalar multn condn fails so

Propn 1 Let $T : V \longrightarrow W$ be linear. Then

a. $T\mathbf{0} =$ b. $T(-\mathbf{v}) =$

Proof:

a.

b.

Thm 1 (Preservation of Lin Combn) Let $T : V \longrightarrow W$ be linear. Then for

scalars

vectors

we have

$T(\lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n) =$

Proof: By induction on n. Just do $n = 2$

case here

$T(\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2) = T(\lambda_1 \mathbf{v}_1) +$

Remark Thm $\Longrightarrow$ if you know $T : V \longrightarrow W$ is linear and the values of T on a spanning set for V then T is determined.

To illustrate this,

E.g. 6 Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ linear &

$T(1,1)^T = (1,1,1)^T, T(1,0)^T = (2,2,1)^T$.

Note $(1,1)^T, (1,0)^T$ span

Then

E.g. 7 Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ linear and

$T(1,0)^T = (1,2)^T, T(0,1)^T = (3,4)^T.$

Then $T(\lambda, \mu)^T = T(\lambda(1,0)^T +$

$= \lambda T(1,0)^T +$

$= \lambda$

$=$

In fact, any linear $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is multn

by

Thm 2 (Matrix Representation Thm)

Let $T : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ be linear.

$\mathbf{e}_1, \ldots, \mathbf{e}_n \in$

Define $\mathbf{a}_1 = T\,\mathbf{e}_1, \ldots$

& $A =$

Then $T = T_A$ i.e.

Proof: $T(x_1, \ldots, x_n)^T = T(x_1\,\mathbf{e}_1 +$

$= T(x_1\,\mathbf{e}_1) +$

$= x_1 T$

$= x_1$

Visualising Linear Transformations

Consider $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ linear with

$T\,\mathbf{e}_1 = \mathbf{a}_1, T\,\mathbf{e}_2 = \mathbf{a}_2$

$$T \begin{pmatrix} 1 \\ 1 \end{pmatrix} =$$

Note grid lines go to

E.g. 8 Scaling axes

$$A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Let $T = T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$