Lecture 8: Complex polynomials

Aim Lecture There is a rich theory associated to complex polynomials because

Defn A complex polynomial is an expression

Let $\mathbb{C}[z]$ denote

If all $a_i \in$

Define the degree

To a complex (or real) polynomial we obtain a function p :

The reason complex numbers are so important is the following

Fundamental Thm of Algebra (Gauss)
Any complex polynomial has a (

Idea of Proof Easy case: real polynomial odd degree e.g. $p(z) = z^5 - 3z + 5$.

Suppose now in general case $p(z) = a_n z^n +$

$\ldots a_1 z + a_0,\ a_n \neq$

(SEE MAPLE e.g.) Let $C_r := \{z||z| = r\}$.

$p(C_r) := \{p(z)||z| = r\} =$

We want $p(C_r)$ to go through 0 for some r.

If r big

p

$p(z) =$

dominated by

This is

Now shrink r. "n-fold circle" $p(C_r)$

Remainder Thm Let $p(z)$ be a poly &
$\alpha \in$ The remainder

Proof If quotient is $q(z)$ &

An immediate corollary is

Factor Thm Let $p(z)$ be a poly & $\alpha \in$

Then $z - \alpha$

Thm Let $p(z) = a_n z^n + \ldots + a_1 z + a_0$ be a complex poly. We can

$$(*) \qquad p(z) = a_n($$

where $\alpha_1,$

Furthermore, the factorisation in (*) is unique up to permuting factors.

Defn The number of times a root α_i occurs in the factorisation is called the multiplicity of the root.

Proof thm Clear when $\deg p(z) =$

Fund thm $\implies$

factor thm $\Longrightarrow$

Now $q(z)$ is a poly with leading coeff

$q(z) =$

$p(z) =$

For $\alpha \in \mathbb{C}$, $p(\alpha) =$

so α is a root iff

If α is a root then $z - \alpha$ divides $p(z)$ and uniqueness of factorn follows from uniqueness of factorn for

e.g.1 Factorise $z^2 + 4$.

Corollary a) If poly $p(z), q(z)$ have degrees $\leq n$ & agree on

b) Any 2 poly which agree on an infinite set

Proof Clear a) $\implies$ b). To prove a), suppose to contrary $p(z) \neq q(z)$. Let $m :=$ $\deg p(z) -$

$p(z) - q(z) = a_m($

Pick

e.g. 2 Given 3 distinct points $(x_1, y_1), (x_2, y_2)$ & (x_3, y_3), there is at most 1 parabola of the form $y = p(x)$ going through those points.

Why? If $y = q(x)$ also went through those points then

Factorisation over the reals

Propn a) Let $p(z)$ be a real poly & $z = \alpha$

b) $(z - \alpha)(z - \bar{\alpha})$

Proof see Notes §1.10.2 for a). b) clear from any example (see below).

e.g. 3 Factorise $4 - z^6$ i) over $\mathbb{C}$ ii) over $\mathbb{R}$.

i) Find complex roots $z = re^{i\theta}$.

$z^6 =$

cont'd