Lecture 24: Cross Products.

Aim Lecture The study of 3-dim geom is greatly aided by the notion

Warning Cross products only work in

Let $\mathbf{v} = (v_1, v_2, v_3), \mathbf{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$

Defn Define the cross product of

$\mathbf{v} \times \mathbf{w} \overset{\text{fake}}{=}$

$\overset{\text{real}}{=}$

N.B. This product is a vector in

If $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$, we define the triple scalar product of

N.B. This product is a

e.g. 1 a) Find $\mathbf{v} \times \mathbf{w}$ where $\mathbf{v} = (0, 1, 2)$, $\mathbf{w} = (3, 4, 5)$.

b) Find triple scalar product of $\mathbf{u} = (1, -1, 1)$, $\mathbf{v}$ & $\mathbf{w}$.

Ans a)

b)

Laws of Algebra for Cross Product

Propn 1 For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3, \lambda \in \mathbb{R}$.

a) $\mathbf{v} \times \mathbf{v} =$ b) $\mathbf{w} \times \mathbf{v} =$

c) $\mathbf{v} \times \lambda \, \mathbf{w} =$

so in particular $\mathbf{0}$

d) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) =$

Rem a) & c) $\implies$ $\mathbf{v} \times \lambda \, \mathbf{v} = \mathbf{0}$ i.e. parallel

Proof Omitted mostly. Fake defn suggests they follow from properties of det. Actual proofs just involve checking defns. e.g. for c) using above notn

$$\mathbf{v} \times \lambda\, \mathbf{w} = (v_1, v_2, v_3) \times (\lambda w_1, \lambda w_2, \lambda w_3)$$

$$= \left(\begin{vmatrix} v_2 & v_3 \\ \lambda w_2 & \lambda w_3 \end{vmatrix}, \right.$$

$$= \left(\lambda \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, \right.$$

$$=$$

Sim for $\lambda\, \mathbf{v} \times \mathbf{w}$.

Formula for Triple Scalar Product

$$A = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

so that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are

Formula $\mathbf{u}\,.(\mathbf{v} \times \mathbf{w}) =$

Proof

$\mathbf{v} \times \mathbf{w} =$

$\therefore \mathbf{u} . (\mathbf{v} \times \mathbf{w}) = (u_1,$

$= u_1 A_{11} -$

$= |A|$ by expanding

Note 1) Swapping $\mathbf{u}, \mathbf{v}, \mathbf{w}$ around changes scalar triple product by at most a sign $\because$ that's all that

2) $\mathbf{v} . (\mathbf{v} \times \mathbf{w}) =$

i.e. $\mathbf{v} \times \mathbf{w}$ is orthog

Geom Interpretn of Cross Product

Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, & θ be

Propn 2 a) $|\,\mathbf{v} \times \mathbf{w}\,| =$

$=$ area of

b) If $\mathbf{v} \parallel \mathbf{w}$ then

otherwise $\mathbf{v} \times \mathbf{w}$ is

c) this leaves 2 possible dirns for $\mathbf{v} \times \mathbf{w}$. The right

Proof b) is rem to propn 1 & note 2. Proof c) omitted but note it agrees with propn 1b).

Proof a):

Area parallelogram =

It only remains to show other formula which follows from

$(*) \qquad \qquad | \mathbf{v} \times \mathbf{w} |^2 =$

LHS $(*) = (v_2w_3 - v_3w_2)^2 + (v_3w_1 - v_1w_3)^2 + (v_1w_2 - v_2w_1)^2$

RHS $(*) = | \mathbf{v} |^2 | \mathbf{w} |^2 (1-$

$= | \mathbf{v} |^2 | \mathbf{w} |^2 -$

$$= (v_1^2 + v_2^2 + v_3^2)(w_1^2 + w_2^2 + w_3^2) -$$

Expanding shows LHS = RHS. This proves propn.

e.g. 2 $A(1, 2, 3), B(1, 1, 0), C(0, -1, 2)$. Find area $\triangle\ ABC$.

Ans

Area $\triangle\ ABC =$

$\frac{1}{2}$ area parallelogram

sides

$\vec{BA} =$

$\vec{BA} \times \vec{BC} =$

$=$

$\therefore$ Area $\triangle\ ABC = \frac{1}{2}|\vec{BA}$

$=$

Geom Interpretn of Triple Scalar Product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. The parallelepiped P_3 with

$P_3 := \{\lambda\,\mathbf{u} + \mu\,\mathbf{v} + \nu\,\mathbf{w}\,|$

Vol $P_3 =$ (area parallelogram P_2 with

$\times$ (height

height $P_3 = |\,\mathbf{u}\,||\cos$

$=$

Area $P_2 =$

$\therefore$ Vol $P_3 =$

Fact The triple scalar product is, up to sign,

e.g. 3 Find the vol of the parallelepiped with edges going from $O(0,0,0)$ to $A(1,2,3)$, $B(1,1,0)$ & $C(0,-1,2)$.

Ans Volume is $\vec{OA}$.

$$\left|\det\begin{pmatrix}1 & 2 & 3\\ 1 & & \\ & & \end{pmatrix}\right| = \left|\det\begin{pmatrix}1 & 2 & 3\\ & & \\ & & \end{pmatrix}\right| =$$