Lecture 22: The Dot Product of Vectors

Aim Lecture The dot product contains info about

NOTE In this final chapter, $\mathbb{F} = \mathbb{R}$ always.

Length Recall from lecture 9, that for $\mathbf{v} = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$, the length

Defn made to extend defn for geometric vectors. Many obvious facts for geom vectors in $\mathbb{R}^2, \mathbb{R}^3$ hold for

Facts 1) $|\,\mathbf{v}\,| = 0$ iff

2) For $\lambda \in \mathbb{R}$, $|\lambda\,\mathbf{v}\,| =$

Proofs: Easy.

Dot Product Suppose also

$\mathbf{w} = (w_1, \ldots, w_n)^T \in \mathbb{R}^n$.

Defn The dot (or scalar) product of

$\mathbf{v} \,.\, \mathbf{w} :=$

N.B. If you identify 1×1-matrices with scalars then

This clear from any example like

e.g. 1 Let $\mathbf{v} = (1, 2, 3)^T, \mathbf{w} = (4, 5, 6)^T$.

$\mathbf{v} \,.\, \mathbf{w} =$

$\mathbf{v}^T \mathbf{w} =$

Geom Interpretn of Dot Product

Propn Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ where $n = 2$ or 3.

Let θ be the

Then

$(*) \quad \mathbf{v} \,.\, \mathbf{w} =$

Rem Later, we'll use (*) to define angles when $n > 3$.

Proof Cosine law gives

$|\,\mathbf{v} - \mathbf{w}\,|^2 = |\,\mathbf{v}\,|^2 + |\,\mathbf{w}\,|^2 -$

$\therefore |\,\mathbf{v}\,||\,\mathbf{w}\,| \cos\theta =$

$= \frac{1}{2}(\sum a_i^2 +$

e.g. 2 Let $A(0,1), B(1,4), C(3,1)$ & $P(3,3)$. Does P lie on altitude of $\triangle\ ABC$ thru A?

Ans

Suffice show $\vec{AP}$

$\vec{AP} =$

$\vec{AP}.$

Laws of Arithmetic for Dot Product

Propn For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, λ

1) $\mathbf{w} \,.\, \mathbf{v} =$ 2) $\mathbf{v} \,.(\lambda \, \mathbf{w}) =$

3) $\mathbf{u} \,.(\mathbf{v} +$

4) $\mathbf{v} \,.\, \mathbf{v} =$

Proof 1,2,3 follow use matrix "defn".

1) Any 1×1-matrix is

2) $\mathbf{v} \,.(\lambda\,\mathbf{w}) = \mathbf{v}^T\,\lambda\,\mathbf{w} =$

3) $\mathbf{u} \,.(\mathbf{v} + \mathbf{w}) =$

4) $\mathbf{v} \,.\, \mathbf{v} = \sum$

Cauchy-Schwarz Inequality

Thm (Cuachy-Schwarz) Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

Then $|\,\mathbf{v} \,.\, \mathbf{w}\,|$

Equality holds iff

Defn If $\mathbf{v}, \mathbf{w}$ non-zero, thm allows us to define the angle

$\theta :=$

Proof thm Fix $\mathbf{v}, \mathbf{w}$ as in thm. Consider fn of $\lambda \in \mathbb{R}$,

$q(\lambda) = |\,\mathbf{v} - \lambda\,\mathbf{w}\,|^2 = (\mathbf{v} - \lambda\,\mathbf{w}).(\mathbf{v} - \lambda\,\mathbf{w})$

$= \mathbf{v}\,.\,\mathbf{v}$

$= |\,\mathbf{v}\,|^2 -$

Formula for length $\implies$

So real quadratic fn

i.e. $(2\,\mathbf{v}\,.\,\mathbf{w})^2 -$

Note equality holds iff $q(\lambda)$

e.g. 3 If $A(1,0,0,1), B(0,1,0,1), C(1,0,1,0)$.

Find $\theta := \angle ABC$.

Ans $\vec{BA} =$

$\vec{BA}.\vec{BC} =$

$|\vec{BA}| =$

$\cos\theta =$

Minkowski's Triangle Inequality

Thm Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

$|\,\mathbf{v} + \mathbf{w}\,|$

i.e. length of side

of triangle $\leq$

Proof Suffice show

$|\,\mathbf{v} + \mathbf{w}\,|^2 \leq$

$\text{LHS} = (\mathbf{v} + \mathbf{w}).$

$= \mathbf{v} \,.\, \mathbf{v}$

$\leq$

Many traditional facts from 2 & 3-dim geometry can be generalised to n-dimensions e.g. if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are coplanar & arranged as in diagram then