Lecture 21: Determinant of a Square Matrix

Aim Lecture Intro determinant of a square matrix & examine

Apology Most proofs omitted this lecture.

Defn Let $A \in M_{mn}$. We define by induction on n the number $\det A$ or $|A|$ called

$n = 1$: $A = (a)$

$n = 2$: $A =$

Fact See later that $|\det A|$ is the area

Higher dim determinants gen this. For $n \geq 3$ need

Notn Let A_{ij} be the $(n-1) \times (n-1)$-submatrix of A obtained by

$$\text{e.g.} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, A_{23} =$$

The determinant $|A_{ij}|$ is called

Define inductively

$|A| := a_{11}|$

$= \sum_{k=1}^{n} (-1)^{k-1}$

Using this formula called "expanding

e.g. 1 $A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 4 & 2 & 1 \end{vmatrix}$

e.g. 2 We'll show by induction on n that the det of any lower triang matrix

Proof $A =$

If $n = 1$, claim is true. For inductive step, $\det A =$

But A_{11} is lower triang so by inductive hypothesis,

$\therefore \det =$

Propn 2 $\det A = \sum_{k=1}^{n} (-1)^{k-i}$

i.e. you can compute det by "expanding along

Propn 3 $\det A^T =$

Cor: a) Det of upper triang matrix is the

b) Since ^{T} swaps rows & col, you can compute det by "expanding along the j-th i.e. $\det A = \sum_{k=1}^{n} (-1)^{k-j}$

e.g. 3 $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 4 \\ 5 & 7 & 6 \end{pmatrix}$

Expand along

$\det A = -a_{32}$

e.g. 4 If A has a whole row of 0's or whole

Why? Just expand along

Mnemonic How do you remember the sign in front of $a_{kj}|A_{kj}|$ of $a_{ik}|A_{ik}$?

Effect of EROs (ECOs) on det

In following, let A' be matrix obtained by applying some ERO to A.

Propn 4 a) If ERO is row swap Ri

b) If ERO is $Ri \mapsto cRi$

c) If ERO is Ri

Propn 4^T Elem column opern affect det

e.g. 5 If a row (resp col) of A is a scalar multiple of a different

Why? Just apply ERO (resp ECO) to get a

Propn 5 For $A, B \in M_{mn}$,

$\det(AB) =$

Remark If assume propn 5, then it's easy to prove propn 4 as follows.

As in lecture 10, we can write $A' = EA$ where

a) $Ri \leftrightarrow$

b) $Ri \mapsto cRi$

(we didn't check this one but you can check it easily yourself or in §4.3.3 of notes.)

c) $Ri \mapsto Ri+$

Propn 4 follows from propn 5 if we can show

$\det A =$

In cases b),c), this easy $\because$ E is

In case a), this is a simple inductive

Computing det A via EROs

Unless A is special, the most efficient way to compute det A is to apply EROs until A is in

e.g. 6 $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 6 \\ 1 & 1 & 1 & 4 \\ 1 & 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{vmatrix}$$

$=$

Invertibility & det A

Propn 6 a) $A \in M_{nn}$ is invertible iff

b) In this case, $\det A^{-1} =$

Proof a) Apply EROs of form

until you arrive at U which is upper

Note $\det A =$

By lemma 2, lecture 20,

A is invertible $\iff$ all col of

$\iff$ all diag entries

$\iff$ $\det U \neq 0$ $\iff$ $\det A$

b) Propn 5 $\implies$

$(\det A)(\det A^{-1}) = \det$

$\therefore \det A^{-1} =$

Cor Let $A, B \in M_{nn}$ be s.t. AB is invertible. Then so

Proof $0 \neq \det AB =$

Inverse of 2×2-matrix

$A =$

If $\det A \neq 0$ then

$A^{-1} =$

Proof Just multiply the two together.

Remark There's a higher dimensional analogue called Cramer's rule. There too, $|A|$ is in the denominator and the other terms are minors up to sign.