Lecture 20: Inverse of a Matrix

Aim Lecture The notion of inverse allows you to (left & right) "divide" & so is useful

Defn Let $A \in M_{mn}$. A left (resp. right) inverse for A is a matrix B

If B is both a left & right we say A is invertible & B

N.B. If B is a left inverse for A then

BA defined $\implies$

$BA = I$ square $\implies$

i.e. $B \in$

Sim if B is a right

e.g. $A = (1 \quad -1), B = \binom{2}{1}$.

$AB =$

so B is right inverse of A &

But $BA =$

so A not

Uniqueness of Inverse

Propn 1 Let $A \in M_{mn}$ & B_l, B_r be left & right

Then $B_l =$

In particular, the inverse of A is unique & we denote

Proof $B_r = I_n B_r =$

Propn 2 If A is invertible then so is

Proof Write $B = A^{-1}$ so that $AB =$

Defn $\implies A =$

Relevance to solving eqns

For $A \in M_{mn}$ consider

$$A\,\mathbf{x} = \mathbf{b} \ldots (*)$$

1 If B is a left inverse of A:

Suppose there's a soln $\mathbf{x}$ to (*). Then

$B\,\mathbf{b} =$

So soln given uniquely

2 If B is a right inverse of A:

Setting $\mathbf{x} = B\,\mathbf{b}$ gives a

i.e. there exists

3 If A is invertible:

1) & 2) apply so $A\,\mathbf{x} = \mathbf{b}$ always

Calculating Inverses

Saw in lecture 16 that for special A square, solns to $A\,\mathbf{x} = \mathbf{b}$ given uniquely by $\mathbf{x} = C\,\mathbf{b}$ where C is matrix obtained by applying Gauss elim to

$(A|I)$

Following lemmas allow us to identify this C

Lemma 1 Let $C, C' \in M_{mn}$ be such that $C\,\mathbf{b} = C'\,\mathbf{b}$ for

Then C

Proof We'll show columns are

$C\mathbf{e}_i = C'\mathbf{e}_i$ i.e.

Lemma 2 Let $A \in M_{mn}$ & U be a row echelon form for A. The following 4 conds on A are equivalent.

a) A is invertible

b) A is square & all columns of U

c) (A is square &) $A\,\mathbf{x} = \mathbf{b}$ always has

d) A is square & $A\,\mathbf{x} = \mathbf{0}$ has

If in case c) the soln is given by $\mathbf{x} = C$

Proof The equiv of b),c),d) follows from lecture 15 which gives the no. of parameters in soln as

To show a) $\implies$ b) suppose A invertible.
$A\,\mathbf{x} = \mathbf{0}$ has unique soln $\implies$ U has

$\implies m$

Also $A^{-1} \in M_{mn}$ so same argument $\implies$

Let's now prove b) $\implies$ a)

Assuming b), saw in lecture 16, there's a matrix C such that for any $\mathbf{b}$, the unique soln to $A\,\mathbf{x} = \mathbf{b}$ is

$I\,\mathbf{b} = \mathbf{b} = A\,\mathbf{x} =$

Lemma 1 $\implies$

$I\,\mathbf{x} = \mathbf{x} = C$

Lemma 1 $\implies$

So C is

This shows last statement too.

e.g. Find A^{-1} if it's invertible.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 2 \\ 2 & 0 & 5 \end{pmatrix}$$

A

$$(A|I) = \begin{pmatrix} 1 & 0 & 2 & & 1 & 0 & 0 \\ 1 & -1 & 2 & & 0 & 1 & 0 \\ 2 & 0 & 5 & & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & & 1 & 0 & 0 \\ 0 & & & & & 1 & 0 \\ 0 & & & & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & & \end{pmatrix}$$

So A^{-1} is

Propn 3 Let $A, B \in M_{mn}$ be invertible. Then AB

Proof Just check that