Lecture 19: Transpose. Elementary Matrices.

Aim Lecture Examine how transposes allow you to swap results from left

Propn 1 $((A)^T)^T =$

Proof Clear from any e.g.

Propn 2 For matrices A, B & $\lambda \in \mathbb{F}$, the following hold if they make sense.

1) $(A + B)^T =$ 2) $(\lambda A)^T =$

3) $(AB)^T =$

Proof Just compute both sides using defns.

We'll only check 3)

$[(AB)^T]_{ij} = [AB]_{ji} = \sum$

$(AB)^T =$

Applicn We'll show matrix distrib law $L(M+N) = LM + LN$ gives other distrib law $(A+B)C =$

$(A+B)C = (((A+B)C)^T)^T$

$= (C^T(A+B)^T)^T = (C^T(A^T + B^T))^T$

Defn A matrix A is symmetric if

e.g.

N.B. Any symmetric matrix is square.

EROs & Elementary matrices

e.g.

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} =$$

The right multn by the 2×2-matrix above performs an "elem column operation" col 2 $\mapsto$

We generalise as follows.

For $i \neq j$ & $c \in \mathbb{F}$ define $n \times n$-matrix by

$E_c(i, j) =$

i.e. $E_c(i,j)$ looks like I_n except the

Fact Let $A = (\mathbf{a}_1 \ldots \mathbf{a}_n) \in M_{mn}$. Then $AE_c(i,j)$ is the same as A except the

i.e. right multn by $E_c(i,j)$ performs the elem col opern

Proof Let's check j-th col of $AE_c(i,j)$ which is

A

Can check other col sim.

Let's apply transpose to fact with $A = B^T$.

$(B^T E_c(i,j))^T = (E_c(i,j))^T B = E_c(j,$

Since ^{T} swaps columns &

Fact^T Left multn by $E_c(j,i)$ performs the

ERO

Proof Let's check j-th row, others easier.

j-th row $E_c(j,i)B = j$-th col $B^T E_c(i,j)$

$= j$-th col $B^T +$

$= j$-th row $B +$

Factorising Suppose we apply the ERO

A

We can get A' from A by applying the "inverse" ERO

Fact^T $\implies A = E_{-c}($

Row Swaps can also be implemented by matrix multn. For $i \neq j$, let

$E(i, j) :=$

i.e. $E(i, j)$ is like I_n except that

(i, i)-th & (j, j)-th entries

(i, j)-th &

Fact Left multn by $E(i, j)$ performs the ERO

No **proof** It's clear from any e.g.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} =$$

Defn A square matrix A is said to be upper

(resp. lower

$[A]_{ij} =$

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ & 4 & 5 \\ & & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & & \\ 2 & 3 & \\ 4 & 5 & 6 \end{pmatrix}$$

N.B. The transpose of an upper triang matrix is

Matrices of the form $E(i,j), E_c(i,j)$ are called

Propn Any square matrix A can be factorised as

$A = E_1 E_2 \dots E_k A'$

where A' is upper

Proof Clear from facts above &

e.g.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$