Lecture 18: Algebra of Matrices

Aim Lecture You can add & multiply

There are some important differences however.

Notn-Defn Let $M_{mn}(\mathbb{F})$ denote the set of all

In M_{mn} we always have the zero matrix $\mathbf{0}$ which

e.g. in M_{23} zero matrix is

The (i, j)-th entry of $A \in M_{mn}$, denoted

$[A]_{ij}$ is the one in

The size of a matrix in M_{mn} is m by

Addn, subtrn & scalar multn of vectors extends naturally to matrices.

Sum,Difference For $A, B \in$

Define $A \pm B$ to be the $m \times n$-matrix with entries

$[A$

i.e. add or subtract

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 1 \\ -2 & 3 & 4 \end{pmatrix} =$$

Scalar Multn If also $\lambda \in \mathbb{F}$ is a scalar, define $m \times n$-matrix

$[\lambda A]_{ij} :=$

i.e. multiply each entry

e.g.

$$2\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$$

Transpose For $A \in M_{mn}$ define $n \times m$-matrix A^T by

$[A^T]_{ij} :=$

i.e. "reflect" about main diag or swap rows to col **e.g.**

$$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}^T =$$

Matrix Multn Let $B = (\mathbf{b}_1\, \mathbf{b}_2 \ldots \mathbf{b}_p) \in M_{np}$ where $\mathbf{b}_1, \mathbf{b}_2 \ldots, \mathbf{b}_p$ are

Let $\mathbf{x} = (x_1 \ldots x_p)^T$. Recall

$B\,\mathbf{x} = x_1$

Suppose $A \in M_{mn}$ so

no. columns of $A =$

We define the matrix product AB to be the $m \times p$-matrix

$AB = (A$

i.e. columns of AB are

N.B. Recovers old defn when B is

e.g.

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -3 & -2 & 1 \\ 1 & 2 & 3 \end{pmatrix} =$$

N.B.

$$\begin{pmatrix} -3 & -2 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$$

not

Defn important $\because$ of

Thm For matrices $A, B, \mathbf{x}$ as above,

(Assoc Law) $(AB)\,\mathbf{x} =$

In other words if $\mathbf{y}$ is fn of $\mathbf{x}$ given by $\mathbf{y} =$ $B\,\mathbf{x}$ and $\mathbf{z}$ is a fn of $\mathbf{y}$

$\mathbb{F}^p \longrightarrow$

$\mathbf{x}$

Then $\mathbf{z}$ is fn

Proof $A(B\,\mathbf{x}) = A(x_1\,\mathbf{b}_1 +$

$= A(x_1\,\mathbf{b}_1) +$

$= x_1($

$= (AB)\,\mathbf{x}$

Formula $[AB]_{ij} = \sum_{k=1}^{n}$

Proof $[AB]_{ij} = i$-th entry

Rem Formula useful for proving results about matrix multn.

Laws of Arithmetic for Matrices

A, B, C below are matrices & $\lambda, \mu \in \mathbb{F}$ scalars

The following formulae hold whenever they are defined.

1) (Assoc) $(AB)C =$

2) (Distrib) $A(B + C) =$

3) (Distrib) $(A + B)C =$

4) $A(\lambda B) =$

5) (Identity) $AI = A$ &

6) $\lambda(A+$

7) $(\lambda+$

Proofs We've seen 1,2,4 when B and / or C is a vector. Gen case follows from this

e.g. if i-th col of C is $\mathbf{c}_i$ then

i-th col of $(AB)C =$

$\Longrightarrow$ (AB)C =

Sim for 2 & 4. Proof of 3 next lecture

5 is easy computn e.g. if $A \in M_{32}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}, I = I_2$$

$$AI = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

For 6 & 7, note 4 & 5 $\implies \lambda A =$

so 2 $\implies$

Failure of Commutativity & Consequences

e.g. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For these A, B, we can simplify $(A + B)^2$ using the relation $AB =$